

Evolution Semigroups in Supersonic Flow-Plate Interactions

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Abstract

We consider the well-posedness of a model for a flow-structure interaction. This model describes the dynamics of an elastic flexible plate with clamped boundary conditions immersed in a supersonic flow. A perturbed wave equation describes the flow potential. The plate's out-of-plane displacement can be modeled by various nonlinear plate equations (including von Karman and Berger). Supersonic regimes corresponding to the flow provide for new mathematical challenge that is related to the loss of ellipticity in a stationary dynamics. This difficulty is present also in the linear model. We show that the linearized model is well-posed on the state space (as given by finite energy considerations) and generates a strongly continuous semigroup. We make use of these results along with sharp regularity of Airy's stress function (obtained by compensated compactness method) to conclude global-in-time well-posedness for the fully nonlinear model.

The proof of generation has two novel features, namely: (1) we introduce a new flow potential velocity-type variable which makes it possible to cover both subsonic and supersonic cases, and to split the dynamics generating operator into a skew-adjoint component and a perturbation acting *outside of the state space*. Performing semigroup analysis also requires a nontrivial approximation of the domain of the generator. The latter is due to the loss of ellipticity. And (2) we make critical use of *hidden trace regularity* for the flow component of the model (in the abstract setup for the semigroup problem) which allows us to develop a fixed point argument and eventually conclude well-posedness. This well-posedness result for supersonic flows (in the absence of regularizing rotational inertia) has been hereto open. The use of semigroup methods to obtain well-posedness opens this model to long-time behavior considerations.

Key terms: flow-structure interaction, nonlinear plate, supersonic and subsonic flows, nonlinear semigroups, well-posedness, dynamical systems.

1 Introduction

1.1 Physical Motivation

The interaction of a thin, flexible structure with a surrounding flow of gas is one of the principal problems in aeroelasticity. Models of this type arise in many engineering applications such as studies of bridges and buildings in response to wind, snoring and sleep apnea in the human palate, and in the stability and control of wings and aircraft structures [1, 6, 21, 24, 34]. In general, for an abstract setup, we aim to model the oscillations of a thin flexible structure interacting with

an inviscid potential flow in which it is immersed. These models accommodate certain physical parameters, but one of the key parameters is the flow velocity of the unperturbed flow of gas.

Specifically, we deal with a common flow-structure PDE model which describes the interactive dynamics between a (nonlinear) plate and the surrounding potential flow (see, e.g., [7, 22]). This model is one of the standard models in the applied mathematics literature for the modeling of flow-structure interactions (see, e.g., [7, 22] and also [20, 21] and the references therein).

The main goal of this paper is to present Hadamard well-posedness results for the model in the presence of *supersonic flow velocities*. While *subsonic* flows have received recent attention which has resulted in a rather complete mathematical theory of well-posedness [8, 10, 16, 17, 44] and spectral behavior for reduced (linear) models [3, 39], this is not the case for the *supersonic flow velocities*¹. The mathematical difficulty in going from a subsonic to supersonic regimes is apparent when one inspects the *formal* energy balance. There is an apparent *loss of ellipticity* affecting the static problem. This, in turn, leads to the appearance of boundary trace terms that can not be handled by known (elliptic) PDE-trace theories. Successful handling of this issue yields new methodology which is based on appropriate (microlocal) boundary trace estimates and effectively compensates for this loss of ellipticity. The method here presented additionally covers (with minimal adjustments) subsonic flows.

Thus, with respect to the supersonic model, this paper addresses the open question of well-posedness of *finite energy solutions*, which is the most fundamental for future studies. Well-posedness results are necessary mathematically in order to begin long-time behavior and control studies of the model, which belong to the most interesting and pertinent mathematical studies in application for PDE models. Having shown well-posedness allows us to move into stability studies in the presence of control mechanisms [28, 30].

1.2 Notation

For the remainder of the text we write \mathbf{x} for $(x, y, z) \in \mathbb{R}_+^3$ or $(x, y) \in \Omega \subset \mathbb{R}_{\{(x,y)\}}^2$, as dictated by context. Norms $\|\cdot\|$ are taken to be $L_2(D)$ for the domain dictated by context. Inner products in $L_2(\mathbb{R}_+^3)$ are written (\cdot, \cdot) , while inner products in $L_2(\mathbb{R}^2 \equiv \partial\mathbb{R}_+^3)$ are written $\langle \cdot, \cdot \rangle$. Also, $H^s(D)$ will denote the Sobolev space of order s , defined on a domain D , and $H_0^s(D)$ denotes the closure of $C_0^\infty(D)$ in the $H^s(D)$ norm which we denote by $\|\cdot\|_{H^s(D)}$ or $\|\cdot\|_{s,D}$. We make use of the standard notation for the trace of functions defined on \mathbb{R}_+^3 , i.e. for $\phi \in H^1(\mathbb{R}_+^3)$, $\gamma[\phi] = \phi|_{z=0}$ is the trace of ϕ on the plane $\{\mathbf{x} : z = 0\}$.

1.3 PDE Description of the Model

The model in consideration describes the interaction between a nonlinear plate with a field or flow of gas above it. To describe the behavior of the gas we make use of the theory of potential flows (see, e.g., [7, 20] and the references therein) which produce a perturbed wave equation for the velocity potential of the flow. The oscillatory behavior of the plate is governed by the second order (in time) Kirchhoff plate equation with a general nonlinearity. We will consider certain ‘physical’ nonlinearities which are used in the modeling of the large oscillations of thin, flexible plates - so-called *large deflection theory*.

The environment we consider is $\mathbb{R}_+^3 = \{(x, y, z) : z \geq 0\}$. The plate is modeled by a bounded domain $\Omega \subset \mathbb{R}_{\{(x,y)\}}^2 = \{(x, y, z) : z = 0\}$ with smooth boundary $\partial\Omega = \Gamma$. The plate is embedded in a ‘large’ rigid body (producing the so-called *clamped* boundary conditions) immersed in an inviscid flow (over body) with velocity $U \neq 1$ in the negative x -direction². This

¹We exclude the models which are based on the so-called “piston” theory, see [7, Chapter 4], [22, Part I], and also Remark 6.2.2 in [16] for a recent discussion.

²Here we normalize $U = 1$ to be Mach 1, i.e. $0 \leq U < 1$ is subsonic and $U > 1$ is supersonic.

situation corresponds to the dynamics of a panel element of an aircraft flying with the speed U , see, e.g., [21].

The scalar function $u : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ represents the vertical displacement of the plate in the z -direction at the point $(x; y)$ at the moment t . We take the nonlinear Kirchhoff type plate with clamped boundary conditions³:

$$\begin{cases} u_{tt} + \Delta^2 u + f(u) = p(\mathbf{x}, t) & \text{in } \Omega \times (0, T), \\ u(0) = u_0; \quad u_t(0) = u_1, \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (1.1)$$

The aerodynamical pressure $p(\mathbf{x}, t)$ represents the coupling with the flow and will be given below.

In this paper we consider a general situation that covers typical nonlinear (cubic-type) force terms $f(u)$ resulting from aeroelasticity modeling [7, 20, 21, 25]. These include:

Assumption 1.1. 1. *Kirchhoff model:* $u \mapsto f(u)$ is the Nemytski operator with a function $f \in \text{Lip}_{\text{loc}}(\mathbb{R})$ which fulfills the condition

$$\liminf_{|s| \rightarrow \infty} \frac{f(s)}{s} > -\lambda_1, \quad (1.2)$$

where λ_1 is the first eigenvalue of the biharmonic operator with homogeneous Dirichlet boundary conditions.

2. *Von Karman model:* $f(u) = -[u, v(u) + F_0]$, where F_0 is a given function from $H^4(\Omega)$ and the von Karman bracket $[u, v]$ is given by

$$[u, v] = \partial_x^2 u \cdot \partial_y^2 v + \partial_y^2 u \cdot \partial_x^2 v - 2 \cdot \partial_{xy}^2 u \cdot \partial_{xy}^2 v,$$

and the Airy stress function $v(u)$ solves the following elliptic problem

$$\Delta^2 v(u) + [u, u] = 0 \quad \text{in } \Omega, \quad \partial_\nu v(u) = v(u) = 0 \quad \text{on } \partial\Omega. \quad (1.3)$$

Von Karman equations are well known in nonlinear elasticity and constitute a basic model describing nonlinear oscillations of a plate accounting for large displacements, see [26] and also [16, 18] and references therein.

3. *Berger Model:* In this case the feedback force f has the form

$$f(u) = \left[\kappa \int_\Omega |\nabla u|^2 dx - \Gamma \right] \Delta u,$$

where $\kappa > 0$ and $\Gamma \in \mathbb{R}$ are parameters, for some details and references see [5] and also [13, Chap.4].

For the flow component of the model, we make use of linearized potential theory, and we know [6, 7, 21] that the (perturbed) flow potential $\phi : \mathbb{R}_+^3 \rightarrow \mathbb{R}$ must satisfy the perturbed wave equation below (note that when $U = 0$ this is the standard wave equation):

$$\begin{cases} (\partial_t + U \partial_x)^2 \phi = \Delta \phi & \text{in } \mathbb{R}_+^3 \times (0, T), \\ \phi(0) = \phi_0; \quad \phi_t(0) = \phi_1, \\ \partial_\nu \phi = d(\mathbf{x}, t) & \text{on } \mathbb{R}_{\{(x, y)\}}^2 \times (0, T). \end{cases} \quad (1.4)$$

³While being the most physically relevant boundary conditions for the flow-plate model, clamped boundary conditions allow us to avoid certain technical issues in the consideration and streamline our exposition. Other possible and physically pertinent plate boundary conditions in this setup include: hinged, hinged dissipation, and combinations thereof [27].

Here, without loss of generality, the density of an perturbed airflow is assumed to be equal 1. The strong coupling in the model takes place in the downwash term of the flow potential (the Neumann boundary condition) by taking

$$d(\mathbf{x}, t) = -[(\partial_t + U\partial_x)u(\mathbf{x})] \cdot \mathbf{1}_\Omega(\mathbf{x})$$

and by taking the aerodynamical pressure of the form

$$p(\mathbf{x}, t) = (\partial_t + U\partial_x)\gamma[\phi] \quad (1.5)$$

in (1.1) above. This gives the fully coupled model:

$$\begin{cases} u_{tt} + \Delta^2 u + f(u) = (\partial_t + U\partial_x)\gamma[\phi] & \text{in } \Omega \times (0, T), \\ u(0) = u_0; \quad u_t(0) = u_1, \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega \times (0, T), \\ (\partial_t + U\partial_x)^2 \phi = \Delta \phi & \text{in } \mathbb{R}_+^3 \times (0, T), \\ \phi(0) = \phi_0; \quad \phi_t(0) = \phi_1, \\ \partial_\nu \phi = -[(\partial_t + U\partial_x)u(\mathbf{x})] \cdot \mathbf{1}_\Omega(\mathbf{x}) & \text{on } \mathbb{R}_{\{(x,y)\}}^2 \times (0, T). \end{cases} \quad (1.6)$$

1.3.1 Parameters and New Challenges

We do not include the full rotational inertia term $M_\alpha = (1 - \alpha\Delta)u_{tt}$ in the LHS of plate equation, i.e., we take $\alpha = 0$. In some considerations (see [27]) this term is taken to be proportional to the cube of the thickness of the plate, however, it is often neglected in large deflection theory. From the mathematical point of view, this term is regularizing in that it provides additional smoothness for the plate velocity u_t , i.e. $L_2(\Omega) \rightarrow H^1(\Omega)$. This is a key mathematical assumption in our analysis which separates it from previous supersonic considerations and increases the difficulty of the analysis. This will be further elaborated upon in the discussion of previous literature below. The case $\alpha > 0$ presents modeling difficulties in problems with flow coupling interface, but is often considered as a preliminary step in the study limiting problems as $\alpha \searrow 0$ (see [9, 10], [31], and also [16]) where subsonic regimes were studied. On the other hand, the case when rotational terms are not included ($\alpha = 0$) leads to substantial new mathematical difficulties due to the presence of flow trace terms interacting with the plate.

The second key parameter is the unperturbed flow velocity U . Here we take $U \neq 1$ arbitrary. However, the supersonic case ($U > 1$) is the most interesting case from the point of view of application and engineering. Results in this case can be more challenging, due to the loss of strong ellipticity of the spatial flow operator in (1.4). For the subsonic case ($0 \leq U < 1$) there are other methods available, see, e.g., [8, 16, 17, 44]. However, for the *non-rotational* case $\alpha = 0$ in the *supersonic regime* $U > 1$, the problem of well-posedness of finite energy solutions is challenging and *has been hereto open*. As is later expounded upon, the lack of sufficient differentiability and compactness for the plate velocity component u_t renders the existing methods (see [16, Sections 6.5 and 6.6], for instance) inapplicable.

The aim of this paper is to provide an affirmative answer to the well-posedness question in the (mathematically) most demanding case with $\alpha = 0$ and $U > 1$. In fact, we will show that the resulting dynamics generate a *nonlinear semigroup associated with mild solutions*. Though in this treatment we focus on the most challenging case: $\alpha = 0$ and $U > 1$, the new methods developed apply to the full range $U \neq 1$.

1.4 Energies and State Space

In the subsonic case $0 \leq U < 1$ energies can be derived by applying standard multipliers u_t and ϕ_t along with boundary conditions to obtain the energy relations for the plate and the flow.

This procedure leads to the energy which is bounded from below in the subsonic case. However, it is apparent in the *supersonic* case that we will obtain an unbounded (from below) energy of the flow. Hence, we instead make use of the flow acceleration multiplier $(\partial_t + U\partial_x)\phi \equiv \psi$. Our so-called change of variable is then $\phi_t \rightarrow (\phi_t + U\phi_x) = \psi$. Thus for the flow dynamics, instead of $(\phi; \phi_t)$ we introduce the phase variables $(\phi; \psi)$.

We then have a new description of our coupled system as follows:

$$\begin{cases} (\partial_t + U\partial_x)\phi = \psi & \text{in } \mathbb{R}_+^3 \times (0, T), \\ (\partial_t + U\partial_x)\psi = \Delta\phi & \text{in } \mathbb{R}_+^3 \times (0, T), \\ \partial_\nu\phi = -[(\partial_t + U\partial_x)u(\mathbf{x})] \cdot \mathbf{1}_\Omega(\mathbf{x}) & \text{on } \mathbb{R}_{\{(x,y)\}}^2 \times (0, T), \\ u_{tt} + \Delta^2 u + f(u) = \gamma[\psi] & \text{in } \Omega \times (0, T), \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (1.7)$$

This leads to the following (formal) energies, arrived at via Green's Theorem:

$$\begin{aligned} E_{pl}(t) &= \frac{1}{2}(\|u_t\|^2 + \|\Delta u\|^2) + \Pi(u), \\ E_{fl}(t) &= \frac{1}{2}(\|\psi\|^2 + \|\nabla\phi\|^2), \\ \mathcal{E}(t) &= E_{pl}(t) + E_{fl}(t), \end{aligned} \quad (1.8)$$

where $\Pi(u)$ is a potential of the nonlinear force $f(u)$, i.e. we assume that $f(u)$ is a Fréchet derivative of $\Pi(u)$, $f(u) = \Pi'(u)$. Hypotheses concerning $\Pi(u)$ are motivated by the examples described in Assumption 1.1 and will be given later (see the statement of Theorem 3.10).

With these energies, we have the formal energy relation⁴

$$\mathcal{E}(t) + U \int_0^t \langle u_x, \gamma[\psi] \rangle dt = \mathcal{E}(0). \quad (1.9)$$

This energy relation provides the first motivation for viewing the dynamics (under our change of phase variable) as comprised of a generating piece and a perturbation.

Finite energy constraints manifest themselves in the natural requirements on the functions ϕ and u :

$$\phi(\mathbf{x}, t) \in C(0, T; H^1(\mathbb{R}_+^3)) \cap C^1(0, T; L_2(\mathbb{R}_+^3)), \quad (1.10)$$

$$u(\mathbf{x}, t) \in C(0, T; H_0^2(\Omega)) \cap C^1(0, T; L_2(\Omega)). \quad (1.11)$$

In working with well-posedness considerations (and thus dynamical systems), the above finite energy constraints lead to the so-called finite energy space, which we will take as our state space:

$$Y = Y_{fl} \times Y_{pl} \equiv (H^1(\mathbb{R}_+^3) \times L_2(\mathbb{R}_+^3)) \times (H_0^2(\Omega) \times L_2(\Omega)). \quad (1.12)$$

Remark 1.1. The energy $E_{fl}(t)$ defined above coincides with the energy $E_{fl}^{(2)}(t)$ introduced in [9, 10], see also [16, Sect.6.6]. As previously indicated, in the subsonic case, the standard flow multiplier ϕ_t is used in the analysis, rather than $\psi = \phi_t + U\phi_x$; this produces differing flow energies and an interactive term $E_{int}(t)$, which does not appear here (see, e.g., [8, 16, 17, 44]). More specifically, the flow component of the energy in this case is given by

$$E_{fl}^{(1)}(t) \equiv \frac{1}{2}(\|\phi_t\|^2 + \|\nabla\phi\|^2 - U^2\|\partial_x\phi\|^2)$$

⁴For some details in the rotational inertia case we refer to [10], see also the proof of relation (6.6.4) in [16, Section 6.6].

and the interactive energy $E_{int} = U < \gamma[\phi], \partial_x u >$. The total energy defined as a sum of the three components $\mathcal{E}(t) = E_{fl}^{(1)}(t) + E_{pl}(t) + E_{int}(t)$ satisfies $\mathcal{E}(t) = \mathcal{E}(s)$. We note, that in the *supersonic* case the flow part of the energy $E_{fl}^{(1)}(t)$ is no longer nonnegative. This, being the source of major mathematical difficulties, necessitates a different approach. In fact, the new representation of the energies as in (1.8) provides good topological measure for the sought after solution, however the energy balance is *lost* in (1.9) and, in addition, the boundary term is “leaking energy” and involves the traces of L_2 solutions, which are possibly *not defined* at all.

In view of the above, our strategy will be based on (i) developing theory for the traces of the flow solutions; (ii) counteracting the loss of energy balance relation. The first task will be accomplished by exploiting *sharp* trace regularity in hyperbolic Neumann solutions (see [29, 33, 38, 42] for related results). The second task will benefit critically from the presence of the nonlinearity.

1.5 Definitions of Solutions

In the discussion below, we will encounter strong (classical), generalized (mild), and weak (variational) solutions. In our analysis we will be making use of semigroup theory, hence we will work with *generalized* solutions; these are strong limits of strong solutions. These solutions satisfy an integral formulation of (1.6), and are called *mild* by some authors. In our treatment, we will produce a unique generalized solution, and this, in turn, produces a unique weak solution, see, e.g., [16, Section 6.5.5] and [44].

We now define strong and generalized solutions:

Strong Solutions. A pair of functions $(\phi(x, y, z; t); u(x, y; t))$ satisfying (1.10) and (1.11) is said to be a strong solution to (1.6) on $[0, T]$ if

- $(\phi_t; u_t) \in L^1(a, b; H^1(\mathbb{R}_+^3) \times H_0^2(\Omega))$ and $(\phi_{tt}; u_{tt}) \in L^1(a, b; L_2(\mathbb{R}_+^3) \times L_2(\Omega))$ for any $[a, b] \subset (0, T)$.
- $\Delta^2 u(t) - U\gamma[\partial_x \phi(t)] \in L_2(\Omega)$ (thus $u(t) \in H^{7/2}(\Omega) \cap H_0^2(\Omega)$) and the equation $u_{tt} + \Delta^2 u + f(u) = p(\mathbf{x}, t)$ holds in $H^{-1/2}(\Omega)$ for $t \in (0, T)$ with p given by (1.5).
- $(U^2 - 1)\partial_x^2 \phi(t) - (\partial_y^2 + \partial_z^2)\phi(t) \in L_2(\mathbb{R}_+^3)$ with boundary conditions $\partial_\nu \phi(t) \in H^1(\mathbb{R}^2)$ for all $t \in (0, T)$ and satisfying the relation $\partial_\nu \phi = -[(\partial_t + U\partial_x)u(\mathbf{x})] \cdot \mathbf{1}_\Omega(\mathbf{x})$ on $\mathbb{R}^2 \times (0, T)$. Moreover $(\partial_t + U\partial_x)^2 \phi = \Delta \phi$ holds for almost all $t \in (0, T)$ and $(x, y, z) \in \mathbb{R}_+^3$.
- The initial conditions are satisfied: $\phi(0) = \phi_0$, $\phi_t(0) = \phi_1$, $u(0) = u_0$, $u_t(0) = u_1$.

Remark 1.2. The smoothness properties in the definition above are motivated by the description of the generator of the linear problem in the *supersonic* case $U > 1$ which is given below, see relation (2.2) and Lemma 2.1. In the subsonic case regular solutions display more regularity (see, e.g., [16, Secions 6.4 and 6.5] and [17]). The above analysis also reveals that the degraded differentiability of strong solutions is due to the the loss of ellipticity in the supersonic regime and non-Lopatinski character of the boundary conditions.

As stated above, generalized solutions are strong limits of strong solutions; these solutions correspond to semigroup solutions for an initial datum outside of the domain of the generator.

Generalized Solutions. A pair of functions $(\phi(x, y, z; t); u(x, y; t))$ is said to be a generalized solution to problem (1.6) on the interval $[0, T]$ if (1.11) and (1.10) are satisfied and there exists a sequence of strong solutions $(\phi_n(t); u_n(t))$ with some initial data $(\phi_0^n, \phi_1^n; u_0^n, u_1^n)$ such that

$$\lim_{n \rightarrow \infty} \max_{t \in [0, T]} \left\{ \|\partial_t \phi - \partial_t \phi_n(t)\|_{L_2(\mathbb{R}_+^3)} + \|\phi(t) - \phi_n(t)\|_{H^1(\mathbb{R}_+^3)} \right\} = 0$$

and

$$\lim_{n \rightarrow \infty} \max_{t \in [0, T]} \left\{ \|\partial_t u(t) - \partial_t u_n(t)\|_{L_2(\Omega)} + \|u(t) - u_n(t)\|_{H_0^2(\Omega)} \right\} = 0.$$

We can show that in the case when the nonlinear term f is locally Lipschitz from $H_0^2(\Omega)$ into $L_2(\Omega)$ the generalized solutions are in fact *weak solutions*, i.e., they satisfy the corresponding variational forms (see Definition 6.4.3 in [16, Chapter 6]). This can be verified for strong solutions by straightforward integration with the use of regularity exhibited by strong solutions. Using the (strong) limit definition of generalized solutions, we can pass in the limit and show that the generalized solution satisfies the weak formulation of (1.6). This weak solution is in fact unique - for the proof, we defer to the method presented in [16, Chapter 6].

1.6 Description of Past Results

Flow-structure models have attracted considerable attention in the past mathematical literature, see, e.g., [2, 3, 8, 9, 10, 14, 16, 17, 22, 31, 32, 36, 37, 39, 41, 44] and the references therein. However, the vast majority of the work done have been devoted to numerical and experimental studies, see, e.g., [1, 6, 7, 20, 21, 22, 24] and also the survey [34] and the literature cited there. Much of the studies has been based on linear one-dimensional-special geometries plate models where the goal was to determine the speed at which flutter occurs, see [1, 6, 7, 21, 24, 34] for instance. More recently the study of linear models with a one dimensional structure (beam) and Kutta-Jukovsky boundary conditions found renewed interest and have been extensively pursued in [2, 39, 40, 41]. This line of work has focused on spectral properties of the system, with particular emphasis on identifying aeroelastic eigenmodes corresponding to the associated Possio integral equation.

In contrast, our interest here concerns PDE aspects of the problem, including the fundamental issue of well-posedness of *finite energy* solutions corresponding to nonlinear flow-plate interaction in the principal case for the parameters α and U with clamped plate boundary conditions.

In all parameter cases, one is faced with low regularity of boundary traces due to the failure of Lopatinski conditions (unlike the Dirichlet case [38], where there is no loss of regularity to wave solutions in their boundary traces). In fact, the first contribution to the mathematical analysis of the problem is [9, 10] (see also [16, Section 6.6]), where the case $\alpha > 0$ is fully treated. The method employed in [9, 10, 16] relies on the following main ingredients: (1) sharp microlocal estimates for the solution to the wave equation driven by $H^{1/2}(\Omega)$ Neumann boundary data given by $u_t + Uu_x$. This gives $\phi_t|_\Omega \in L_2(0, T; H^{-1/2}(\Omega))$ [33] (in fact more regularity is presently known: $H^{-1/3}(\Omega)$ [29, 42]); and (2) the regularizing effects on the velocity u_t (i.e. $u_t \in H^1(\Omega)$) rather than just $L_2(\Omega)$), when $\alpha > 0$. The above ingredients, along with an explicit solver for the 3-dimensional wave equation and a Galerkin approximation for the structure allows one to construct a fixed point for the appropriate solution map. The method works well in both cases $0 < U < 1$ and $U > 1$. Similar ideas were used more recently [36, 37] in the case when thermoelastic effects are added to the model; in this case the dynamics also exhibit $H^1(\Omega)$ regularity of the velocity in both the rotational and non-rotational cases due to the analytic regularizing effects induced by thermoelasticity [30]. However, when $\alpha = 0$, and thermoelastic smoothing is not accounted for, there is no additional regularity of u_t beyond $L_2(\Omega)$. In that case the corresponding estimates become singular. This destroys the applicability of previous methods. In summary, much of the work on this problem to date has assumed the presence of additional regularizing terms in the plate equation, or depends critically on the condition $U < 1$. A recent book ([16, Chapter 6]) provides an account of relevant results, including more recent applications of the compactness method in the case $\alpha = 0$ and $0 \leq U < 1$. Existence of a nonlinear semigroup capturing finite energy solutions has been shown in [44], see also [17].

In this treatment we take $\alpha = 0$, corresponding to the more difficult non-rotational model, and we approach the problem with $0 \leq U \neq 1$ from the semigroup point of view - without any reliance on explicit solvers for the flow equation or Galerkin constructions. The advantage of this approach, in addition to solving the fundamental well-posedness question, is the potential for an array of important generalizations, including more general flow equations and more general nonlinearities appearing in the structure. Moreover, it may be possible to view the supersonic global solution we arrive at for $\alpha = 0$ as the uniform limit of solutions as $\alpha \downarrow 0$ (as in the subsonic case [31]).

The main mathematical difficulty of the problem under consideration is the presence of the boundary terms: $(\phi_t + U\phi_x)|_\Omega$ acting as the aerodynamical pressure in the model. When $U = 0$, the corresponding boundary terms exhibit monotone behavior with respect to the energy inner product (see [11], [16, Section 6.2] and [28]) which is topologically equivalent to the topology of the space Y given by (1.12). The latter enables the use of monotone operator theory ([11], [16, Section 6.2] and [28]). However, when $U > 0$ this is no longer true with respect to the topology induced by the energy spaces. The lack of the natural dissipativity for both interface traces, as well as the nonlinear term in the plate equation, make the task of proving well-posedness challenging. In the subsonic case, semigroup methods were applied to the problem by implementing certain bounded adjustments to the inner product structure of the state space which then produced shifted dissipativity [44]. This type of consideration is not possible here, owing to the degeneracy of the standard energy functional when $U \geq 1$.

In contrast to these works, our method and results *do not depend on any smoothing mechanism* (as we take $\alpha = 0$) and are applicable *for all* $U \neq 1$. The key ingredients rely on the development of a suitable trace theory for the velocity of the flow and implementing the corresponding estimates with semigroup theory in extended spaces. In this way obtained a-priori estimates allow for a construction of a nonlinear semigroup which evolves finite energy solutions.

1.7 Statement of Main Results

Recall, our state space for the analysis to follow is

$$Y \equiv H^1(\mathbb{R}_+^3) \times L_2(\mathbb{R}_+^3) \times H_0^2(\Omega) \times L_2(\Omega).$$

Theorem 1.1 (Linear). *Consider linear problem in (1.6) with $f(u) = 0$. Let $T > 0$. Then, for every initial datum $(\phi_0, \phi_1; u_0, u_1) \in Y$ there exists unique generalized solution*

$$(\phi(t), \phi_t(t); u(t), u_t(t)) \in C([0, T], Y) \quad (1.13)$$

which defines a C_0 -semigroup $T_t : Y \rightarrow Y$ associated with (1.7) (where $f = 0$).

For any initial data in

$$Y_1 \equiv \left\{ y = (\phi, \phi_1; u, u_1) \in Y \left| \begin{array}{l} \phi_1 \in H^1(\mathbb{R}_+^3), \quad u_1 \in H_0^2(\Omega), \\ -U^2 \partial_x^2 \phi + \Delta \phi \in L_2(\mathbb{R}_+^3), \\ \partial_\nu \phi = -[u_1 + U \partial_x u] \cdot \mathbf{1}_\Omega \in H^1(\mathbb{R}^2), \\ -\Delta^2 u + U \gamma [\partial_x \phi] \in L_2(\Omega) \end{array} \right. \right\} \quad (1.14)$$

the corresponding solution is also strong.

Theorem 1.2 (Nonlinear). *Let $T > 0$ and let $f(u)$ be any nonlinear internal force (Kirchhoff, von Karman, or Berger) given by Assumption 1.1. Then, for every initial data $(\phi_0, \phi_1; u_0, u_1) \in Y$ there exists unique generalized solution $(\phi, \phi_t; u, u_t)$ to (1.6) possessing property (1.13). This solution is also weak and generates a nonlinear continuous semigroup $S_t : Y \rightarrow Y$ associated with (1.7).*

If $(\phi_0, \phi_1; u_0, u_1) \in Y_1$, where $Y_1 \subset Y$ is given by (1.14), then the corresponding solution is also strong.

Remark 1.3. In comparing the results obtained with a subsonic case, there are two major differences at the qualitative level:

1. Regularity of strong solutions obtained in the subsonic case [17] coincides with regularity expected for classical solutions. In the supersonic case, there is a loss of differentiability in the flow in the tangential x direction, which then propagates to the loss of differentiability in the structural variable u .
2. In the subsonic case one shows that the resulting semigroup is *bounded* in time, see [16, Proposition 6.5.7] and also [17, 44]. This property could not be shown in this analysis, and most likely does not hold. The leak of the energy in energy relation can not be compensated for by the nonlinear terms (unlike the subsonic case).

1.8 Proof Strategy

In order to orient and guide the reader through various stages of the proof, we briefly outline the main steps.

1. As motivated by the linear theory in the subsonic case, we use the modified energy (as given in the previous section) to setup the linear problem abstractly. We decompose the linear dynamics into a dissipative piece \mathbb{A} (unboxed below) and a perturbation piece \mathbb{P} (boxed below):

$$\begin{cases} (\partial_t + U\partial_x)\phi = \psi & \text{in } \mathbb{R}_+^3 \times (0, T), \\ (\partial_t + U\partial_x)\psi = \Delta\phi - \mu\phi & \text{in } \mathbb{R}_+^3 \times (0, T), \\ \partial_\nu\phi = -\partial_t u \cdot \mathbf{1}_\Omega(\mathbf{x}) - \boxed{U\partial_x u \cdot \mathbf{1}_\Omega(\mathbf{x})} & \text{on } \mathbb{R}_{\{(x,y)\}}^2 \times (0, T), \\ u_{tt} + \Delta^2 u = \gamma[\psi] & \text{in } \Omega \times (0, T), \\ u = \partial_\nu u = 0 & \text{in } \partial\Omega \times (0, T). \end{cases} \quad (1.15)$$

We then proceed to show that \mathbb{A} (corresponding to the unboxed dynamics above) is m -dissipative on the state space. While “dissipativity” is natural and built in within the structure of the problem, the difficulty encountered is in establishing the *maximality* property for the generator. The analysis of the the resolvent operator is no longer reducible to strong elliptic theory (unlike the classical wave equation). The “loss of ellipticity” prevents us from using the known tools. To handle this we develop a non-trivial approximation argument to justify the formal calculus; maximality will then be achieved by constructing a suitable bilinear form to which a version of Lax-Milgram argument applies.

2. To handle the “perturbation” of the dynamics, \mathbb{P} (boxed) we cast the problem into an abstract boundary control framework. In order to achieve this, a critical ingredient in the proof is demonstrating “hidden” boundary regularity for the acceleration potential ψ of the flow. It will be shown that this component is an element of a negative Sobolev space $L_2(0, T; H^{-1/2}(\Omega))$. The above regularity allows us to show that the term $\langle u_x, \gamma[\psi] \rangle$ is well-defined via duality. Consequently, the problem with the “perturbation” of the dynamics \mathbb{P} , can be recast as an abstract boundary control problem with appropriate continuity properties of the control-to-state maps.

3. Then, to piece the operators together as $\mathbb{A} + \mathbb{P}$, we make use of variation of parameters with respect to the generation property of \mathbb{A} and appropriate dual setting. This yields an integral equation on the state space (interpreted via duality) which must be formally justified in our abstract framework. This is critically dependent upon point (2.) above. We then run a fixed point argument on the appropriate space to achieve a local-time solution for the fully linear Cauchy problem representing formally the evolution $y_t = (\mathbb{A} + \mathbb{P})y \in Y$. In order to identify its generator, we apply Ball's theorem [4] which then yields global solutions.
4. Lastly, to move to the nonlinear problem, we follow the standard track of writing the nonlinearity as a locally Lipschitz perturbation on the state space Y . In the most difficult case of the von Karman nonlinearity the latter is possible due to the established earlier "sharp" regularity of Airy's stress function [16, pp.44-45]. This allows us to implement local theory with a priori bounds (for T fixed) on the solution. The global a priori bounds result from an appropriate compactness-uniqueness argument supported by a maximum principle applied to Monge-Ampere equation; the latter implies $\|u\|_{L^\infty(\Omega)} \leq C\|u, u\|_{L^1}^{1/2}$ for $u \in H^2(\Omega) \cap H_0^1(\Omega)$, see [16, Sections 1.4 and 1.5, pp. 38-58] for details. The above procedure yields a nonlinear semigroup which, unlike the case of subsonic flow, is not necessarily bounded in time (see the case of subsonic dynamics [16, 17, 44]) and the resolvent of this semigroup is not compact.

2 Abstract Setup

2.1 Operators and Spaces

Define the operator $A = -\Delta + \mu$ with some $\mu > 0^5$ and with the domain

$$\mathcal{D}(A) = \{u \in H^2(\mathbb{R}_+^3) : \partial_\nu u = 0\}.$$

Then $\mathcal{D}(A^{1/2}) = H^1(\mathbb{R}_+^3)$ (in the sense of topological equivalence). We also introduce the standard linear plate operator with clamped boundary conditions: $\mathcal{A} = \Delta^2$ with the domain

$$\mathcal{D}(\mathcal{A}) = \{u \in H^4(\Omega) : u|_{\partial\Omega} = \partial_\nu u|_{\partial\Omega} = 0\} = (H^4 \cap H_0^2)(\Omega).$$

Additionally, $\mathcal{D}(\mathcal{A}^{1/2}) = (H^2 \cap H_0^1)(\Omega)$. Take our state variable to be

$$y \equiv (\phi, \psi; u, v) \in (\mathcal{D}(A^{1/2}) \times L_2(\mathbb{R}_+^3)) \times (\mathcal{D}(\mathcal{A}^{1/2}) \times L_2(\Omega)) \equiv Y.$$

We work with ψ as an independent state variable, i.e., we are not explicitly taking $\psi = \phi_t + U\phi_x$ here.

To build our abstract model, let us define the operator $\mathbb{A} : \mathcal{D}(\mathbb{A}) \subset Y \rightarrow Y$ by

$$\mathbb{A} \begin{pmatrix} \phi \\ \psi \\ u \\ v \end{pmatrix} = \begin{pmatrix} -U\partial_x\phi + \psi \\ -U\partial_x\psi - A(\phi + Nv) \\ v \\ -\mathcal{A}u + N^*A\psi \end{pmatrix} \quad (2.1)$$

⁵We include the term μI in the operator A to avoid a zero point in the spectrum. After we produce our semigroup analysis, we will negate this term in the abstract formulation of the problem in order to maintain the equivalence of the abstract problem and the problem as given in (1.7).

where the Neumann map N is defined as follows:

$$Nf = g \iff (-\Delta + \mu)g = 0 \text{ in } \mathbb{R}_+^3 \text{ and } \partial_\nu g = f \text{ for } z = 0.$$

Properties of this map on bounded domains and \mathbb{R}_+^3 are well-known (see, e.g., [30, p.195] and [16, Chapter 6]), including the facts that $N : H^s(\mathbb{R}^2) \mapsto H^{s+1/2}(\mathbb{R}_+^3)$ and

$$N^*Af = \gamma[f] \text{ for } f \in \mathcal{D}(A),$$

and via density, this formula holds for all $f \in \mathcal{D}(A^{1/2})$ as well. Additionally, when we write Nv for $v : \Omega \rightarrow \mathbb{R}$, we implicitly mean Nv_{ext} , where v_{ext} is the extension⁶ by 0 outside Ω .

The domain of $\mathcal{D}(\mathbb{A})$ is given by

$$\mathcal{D}(\mathbb{A}) \equiv \left\{ y = \begin{pmatrix} \phi \\ \psi \\ u \\ v \end{pmatrix} \in Y \left| \begin{array}{l} -U\partial_x\phi + \psi \in H^1(\mathbb{R}_+^3), \\ -U\partial_x\psi - A(\phi + Nv) \in L_2(\mathbb{R}_+^3), \\ v \in \mathcal{D}(\mathcal{A}^{1/2}) = H_0^2(\Omega), \quad -\mathcal{A}u + N^*A\psi \in L_2(\Omega) \end{array} \right. \right\} \quad (2.2)$$

We can further characterize the domain:

Since on $\mathcal{D}(\mathbb{A})$ we have that $\psi = U\partial_x\phi + h$ for some $h \in H^1(\mathbb{R}_+^3)$, then $\gamma[\psi] \in H^{-1/2}(\mathbb{R}^2)$ (we identify \mathbb{R}^2 and $\partial\mathbb{R}_+^3$). This implies $\gamma[\psi]|_\Omega \in H^{-1/2}(\Omega) = [H_0^{1/2}(\Omega)]'$. Therefore we have that $\mathcal{A}u \in H^{-1/2}(\Omega) \subset [\mathcal{D}(\mathcal{A}^{1/8})]'$ (recall that by interpolation the relation $\mathcal{D}(\mathcal{A}^{1/8}) \subset H_0^{1/2}(\Omega)$ holds). Thus

$$u \in \mathcal{D}(\mathcal{A}^{7/8}) \subset H^{7/2}(\Omega).$$

Moreover, for smooth functions $\tilde{\psi} \in L_2(\mathbb{R}_+^3)$ we have that

$$(U\partial_x\psi + A\phi, \tilde{\psi})_{L_2(\mathbb{R}_+^3)} = (U\partial_x\psi + A(\phi + Nv), \tilde{\psi})_{L_2(\mathbb{R}_+^3)} - \langle v, \gamma[\tilde{\psi}] \rangle_{L_2(\mathbb{R}^2)}.$$

Thus on the account that $(\phi, \psi; u, v) \in \mathcal{D}(\mathbb{A})$, so that $U\partial_x\psi - A(\phi + Nv) \in L_2(\mathbb{R}_+^3)$ and $v \in H_0^2(\Omega)$ we have that

$$|(U\partial_x\psi + A\phi, \tilde{\psi})_{\mathbb{R}_+^3}| \leq C\|\tilde{\psi}\|_{\mathbb{R}_+^3} + \|\gamma[\tilde{\psi}]\|_{H^{-2}(\mathbb{R}^2)}$$

for any $\tilde{\psi} \in L_2(\mathbb{R}_+^3)$ with $\gamma[\tilde{\psi}] \in H^{-2}(\mathbb{R}^2)$.

Writing $\Delta\phi = (-U\partial_x\psi) + l_2$ for some $l_2 \in L_2(\mathbb{R}_+^3)$ with the boundary conditions $\partial_\nu\phi = v$, where $v \in H_0^2(\Omega)$, we easily conclude from standard elliptic theory that

$$\phi = -UA^{-1}\partial_x\psi + h_2 \text{ for some } h_2 \in H^2(\mathbb{R}_+^3). \quad (2.3)$$

Substituting this relation into the first condition characterizing the domain in (2.2) we obtain

$$U^2\partial_xA^{-1}\partial_x\psi + \psi = h_1 \in H^1(\mathbb{R}_+^3)$$

which implies

$$U^2\partial_x^2A^{-1}\partial_x\psi + \partial_x\psi \in L_2(\mathbb{R}_+^3)$$

Introducing the variable $p \equiv A^{-1}\partial_x\psi$ one can see that p satisfies wave equation in the supersonic case

$$(U^2 - 1)\partial_x^2p + (-\Delta_{y,z} + \mu)p \in L_2(\mathbb{R}_+^3), \quad (2.4)$$

where $\partial_\nu p = 0$ on the boundary $z = 0$ distributionally.

The observations above lead to the following description of the domain $\mathcal{D}(\mathbb{A})$.

⁶We must utilize this zero extension owing to the structure of the boundary condition for $\partial_\nu\phi$ in (1.15).

Lemma 2.1. *The domain of \mathbb{A} , $\mathcal{D}(\mathbb{A}) \subset Y$, is characterized by: $y \in \mathcal{D}(\mathbb{A})$ implies*

- $y = (\phi, \psi, u, v) \in Y$, $\gamma[\psi] \in H^{-1/2}(\Omega)$,
- $-U\partial_x\phi + \psi \in H^1(\mathbb{R}_+^3)$,
- $v \in \mathcal{D}(\mathcal{A}^{1/2}) = H_0^2(\Omega)$, $u \in \mathcal{D}(\mathcal{A}^{7/8})$,
- $|(-U\partial_x\psi - A\phi, \widehat{\psi})_{\mathbb{R}_+^3}| < \infty$, $\forall \widehat{\psi} \in L_2(\mathbb{R}_+^3)$ with $\gamma[\widehat{\psi}] \in H^{-2}(\mathbb{R}^2)$,
- $U^2\partial_x^2 A^{-1}\partial_x\psi + \partial_x\psi \in L_2(\mathbb{R}_+^3)$ or (2.4) holds. Since by (2.3) $\phi = -Up + h_2$ for some $h_2 \in H^2(\mathbb{R}_+^3)$, equation (2.4) can be also written explicitly in terms of ϕ as

$$(U^2 - 1)\partial_x^2\phi + (-\Delta_{y,z} + \mu)\phi \in L_2(\mathbb{R}_+^3)$$

where $\partial_\nu\phi = v_{ext}$ on the boundary $z = 0$.

2.2 Cauchy Problem and Unbounded Perturbation in Extended Space

With this setup, we will be in a position to show that \mathbb{A} is m -dissipative. A peculiar feature introduced by the presence of the supersonic parameter is the loss of uniform ellipticity in the static version of the perturbed wave operator and the loss of compactness in the resolvent operator. The domain of \mathbb{A} does not possess sufficient regularity. To cope with this difficulty, suitable approximation of the domain will be introduced. In view of this, the proof of the maximality property is involved here. The obtained result will give that the Cauchy problem

$$y_t = \mathbb{A}y, \quad y(0) = y_0 \in Y \tag{2.5}$$

is well-posed on Y . We will then consider the (semigroup) perturbation

$$\mathbb{P} \begin{pmatrix} \phi \\ \psi \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -UAN\partial_x u \\ 0 \\ 0 \end{pmatrix}.$$

The issue here is the unbounded “perturbation”, which does not reside in the state space Y . Note that $\mathcal{R}\{AN\} \not\subset L_2(\mathbb{R}_+^3)$, and only the trivial element 0 is in the domain of AN when the latter is considered with the values in Y . This fact forces us to construct a perturbation theory which operates in extended (dual) spaces. This step will rely critically on “hidden” boundary regularity of the acceleration potential ψ - established in the next section. As a consequence, we show that the resulting Cauchy problem $y_t = (\mathbb{A} + \mathbb{P})y$, $y(0) = y_0 \in Y$ yields well-posedness for the full system in (1.15). Application of Ball’s theorem [4] allows us to conclude that $\mathbb{A} + \mathbb{P}$, with an appropriately defined domain, is a generator of a strongly continuous semigroup on Y .

3 Proof of Main Result

3.1 Hidden Regularity of $\gamma[\psi]$

We consider the following initial boundary value problem:

$$\begin{cases} (\partial_t + U\partial_x)^2\phi = \Delta\phi & \text{in } \mathbb{R}_+^3, \\ \phi(0) = \phi_0; \quad \phi_t(0) = \phi_1, & \\ \partial_\nu\phi = h(\mathbf{x}, t) & \text{on } \mathbb{R}_{\{(x,y)\}}^2. \end{cases} \tag{3.1}$$

We assume

$$h(\mathbf{x}, t) \in L_2^{loc}(0, T; L_2(\mathbb{R}^2)). \quad (3.2)$$

We note that initially, in our studies of the partial dynamics associated to the dissipative part of the dynamics (the system in (1.15) with the boxed term removed), we will take $h(\mathbf{x}, t) = -\partial_t u$, where u is a plate component of a generalized solution to the unboxed part of (1.15). Later, in considering the perturbation of the dissipative dynamics, we will take

$$h(\mathbf{x}, t) = -[\partial_t u + U\partial_x \bar{u}]_{ext},$$

where \bar{u} is some other function which belongs the same smoothness class as u . Therefore the requirement in (3.2) is reasonable.

Let ϕ be the energy type solution of (3.1), i.e.

$$(\phi, \phi_t) \in L_2(0, T; H^1(\mathbb{R}_+^3) \times L_2(\mathbb{R}_+^3)), \quad \forall T > 0. \quad (3.3)$$

These solutions exists, at least for sufficiently smooth h (see [33] and [38]).

Our goal is to estimate the trace of the acceleration potential $\phi_t + U\phi_x$ on $z = 0$. The a priori regularity of $\phi(t) \in H^1(\mathbb{R}_+^3)$ implies via trace theory $\phi_x(t)|_{z=0} \in H^{-1/2}(\mathbb{R}^2)$. However, the a priori regularity of ϕ_t does not allow to infer, via trace theory, any notion of a trace. Fortunately we will be able to show that this trace does exist as a distribution and can be measured in a negative Sobolev space. The corresponding result reads:

Lemma 3.1 (Flow Trace Regularity). *Let (3.2) be in force. If $\phi(\mathbf{x}, t)$ satisfies (3.1) and (3.3), then*

$$\partial_t \gamma[\phi], \quad \partial_x \gamma[\phi] \in L_2(0, T; H^{-1/2}(\mathbb{R}^2)) \quad \forall T > 0.$$

Moreover, with $\psi = \phi_t + U\phi_x$ we have

$$\int_0^T \|\gamma[\psi](t)\|_{H^{-1/2}(\mathbb{R}^2)}^2 dt \leq C_T \left(E_{fl}(0) + \int_0^T \|\partial_\nu \phi(t)\|^2 dt \right) \quad (3.4)$$

The above result is critical for the arguments in later portion of this treatment. Specifically, the above result holds for *any* flow solver; we will be applying this result in the case where $\partial_\nu \phi = -v \in C(0, T; L_2(\Omega))$ coming from a semigroup solution generated by \mathbb{A} .

Proof. One can see that the function $\eta(\mathbf{x}, t) = \phi(\mathbf{x} + Ute_1, t) \equiv \phi(x + Ut, y, z, t)$ possesses the same properties as in (3.3) and solves the problem

$$\begin{cases} \partial_t^2 \eta = \Delta \eta & \text{in } \mathbb{R}_+^3, \\ \eta(0) = \phi_0; \quad \eta_t(0) = \phi_1 + U\partial_x \phi_0, \\ \partial_\nu \eta = h^*(\mathbf{x}, t) & \text{on } \mathbb{R}_{\{(x,y)\}}^2, \end{cases}$$

where $h^*(\mathbf{x}, t) = h(\mathbf{x} + Ute_1, t)$ which is also belong to $L_2(0, T; L_2(\mathbb{R}^2))$, $\forall T > 0$.

Our goal is to estimate the time derivative η_t on $z = 0$. Clearly, the a priori regularity of η_t does not allow to infer any notion of trace. However, we will be able to show that

$$\int_0^T \|\gamma[\eta_t](t)\|_{H^{-1/2}(\mathbb{R}^2)}^2 dt \leq C_T \left(\|\eta_0\|_{H^1(\mathbb{R}_+^3)}^2 + \|\eta_1\|^2 + \int_0^T \|h^*(t)\|_{L_2(\mathbb{R}^2)}^2 dt \right). \quad (3.5)$$

Since $\eta_t(\mathbf{x}, t) \equiv \psi(\mathbf{x} + Ute_1, t)$, we can make inverse change of variable and obtain the statement of Lemma 3.1.

In order to prove (3.5) we apply a principle of superposition with respect to the initial and boundary data.

Step 1: Consider $h^* = 0$. Here we apply Theorem 3 in [33] with $k = 0$. This yields

$$\begin{aligned} \|\eta(t)\|_{H^1(\mathbb{R}_+^3)}^2 + \|\eta_t(t)\|^2 + \|\gamma[\eta_t]\|_{L_2(0,T;H^{-1/2}(\mathbb{R}^2))}^2 + \|\gamma[\eta]\|_{L_2(0,T;H^{1/2}(\mathbb{R}^2))}^2 \\ \leq C_T \left[\|\eta(0)\|_{H^1(\mathbb{R}_+^3)}^2 + \|\eta_t(0)\|^2 \right]. \end{aligned} \quad (3.6)$$

Step 2: Consider zero initial data and arbitrary $h^* \in L_2(\Sigma)$ where $\Sigma \equiv (0, T) \times \mathbb{R}^2$. We claim that the following estimate holds:

$$\|\gamma[\eta]\|_{L_2(0,T;H^{-1/2}(\mathbb{R}^2))}^2 + \|\gamma[\eta]\|_{L_2(0,T;H^{1/2}(\mathbb{R}^2))}^2 \leq C_T [\|h^*\|_{L_2(\Sigma)}^2]. \quad (3.7)$$

The proof of the estimate in (3.7) depends on the fact that the problem is defined on a half-space.

Since $\phi_0, \phi_1 = 0$ we then take the Fourier-Laplace transform:

$$t \rightarrow \tau = \xi + i\sigma, \quad (x, y) \rightarrow i\mu = i(\mu_1, \mu_2),$$

with ξ fixed and sufficiently large.

Now, let $\hat{\eta} = \hat{\eta}(z, \mu, \sigma)$ be the Fourier-Laplace transform of η in x, y and t , i.e.

$$\hat{\eta}(z, \mu, \tau) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} dx dy \int_0^{+\infty} dt e^{-\tau t} \cdot e^{-i(x\mu_1 + y\mu_2)} \cdot \eta(x, y, z, t), \quad \text{Re } \tau > 0.$$

This yields the equation

$$\hat{\eta}_{zz} = (|\mu|^2 + \xi^2 - \sigma^2 + 2i\xi\sigma)\hat{\eta},$$

with transformed boundary condition,

$$\hat{\eta}_z(z=0) = \widehat{h^*}(\mu, \tau), \quad \tau = \xi + i\sigma.$$

Solving the ODE in z and choosing the solution which decays as $z \rightarrow +\infty$, we have

$$\hat{\eta}(z, \mu, \tau) = -\frac{1}{\sqrt{s}} \widehat{h^*}(\mu, \tau) \exp(-z\sqrt{s}),$$

with

$$s \equiv |\mu|^2 + \tau^2 = (|\mu|^2 - \sigma^2 + \xi^2) + 2i\xi\sigma;$$

where the square root \sqrt{s} is chosen such that $\text{Re}\sqrt{\tau^2 + |\mu|^2} > 0$ when $\text{Re } \tau > 0$. On the boundary $z = 0$ we have $\hat{\eta}(z=0, \mu, \tau) = \frac{1}{\sqrt{s}} \widehat{h^*}(\mu, \tau)$. Taking the time derivative amounts to premultiplying by $\tau = \xi + i\sigma$, and with a slight abuse of notation, we have

$$\hat{\eta}_t(z=0, \mu, \tau) = \frac{\tau}{\sqrt{s}} \widehat{h^*}(\mu, \tau), \quad \tau = \xi + i\sigma.$$

Denoting the multiplier $\frac{\xi + i\sigma}{\sqrt{s}} \equiv m(\xi, \sigma, \mu)$, we can infer the trace regularity of η from $m(\xi, \sigma, \mu)$:

$$\begin{aligned} |m(\xi, \sigma, \mu)| &= \left| \frac{\xi + i\sigma}{\sqrt{s}} \right| = \frac{\sqrt{\xi^2 + \sigma^2}}{(|s|)^{1/4}} \\ &= \frac{\sqrt{\sigma^2 + \xi^2}}{((|\mu|^2 - \sigma^2)^2 + \xi^4 + 2\xi^2\sigma^2 + 2|\mu|^2\xi^2)^{1/4}}. \end{aligned}$$

Lemma 3.2. *We have the following estimate*

$$|m(\xi, \sigma, \mu)| \leq 2 \left[1 + \frac{|\mu|^2}{\xi^2} \right]^{1/4} \quad \forall (\xi; \sigma, \mu) \in \mathbb{R}^4, \quad \xi \neq 0. \quad (3.8)$$

Proof. First, we can take each of the arguments ξ , σ and $|\mu|$ to be positive. Next we consider the partition of the first quadrant of the $(|\mu|, \sigma)$ -plane as follows:

$$\begin{cases} (a) & |\mu| \leq \sigma/\sqrt{2}, \\ (b) & |\mu| \geq \sqrt{2}\sigma, \\ (c) & \sigma/\sqrt{2} < |\mu| < \sqrt{2}\sigma. \end{cases}.$$

In cases (a) and (b) above, we can write

$$|m(\xi, \sigma, \mu)| \leq \frac{\sqrt{\sigma^2 + \xi^2}}{\left(||\mu|^2 - \sigma^2|^2 + \xi^4 \right)^{1/4}} \leq \frac{\sqrt{\sigma^2 + \xi^2}}{\left(\sigma^4/4 + \xi^4 \right)^{1/4}} \leq 2.$$

In case (c) we have

$$\begin{aligned} |m(\xi, \sigma, \mu)| &\leq \frac{\sqrt{\sigma^2 + \xi^2}}{(\xi^4 + 2\xi^2\sigma^2 + 2|\mu|^2\xi^2)^{1/4}} \leq \frac{1}{|\xi|^{1/2}} \frac{\sqrt{\sigma^2 + \xi^2}}{(\xi^2 + 2\sigma^2 + 2|\mu|^2)^{1/4}} \\ &\leq \frac{1}{|\xi|^{1/2}} \frac{\sqrt{\sigma^2 + \xi^2}}{(\xi^2 + 3\sigma^2)^{1/4}} \leq \frac{1}{|\xi|^{1/2}} (\sigma^2 + \xi^2)^{1/4} \leq \frac{1}{|\xi|^{1/2}} (2|\mu|^2 + \xi^2)^{1/4}. \end{aligned}$$

This implies the estimate in (3.8). \square

By the inverse Fourier-Laplace transform we have that

$$e^{-\xi t} \eta_t(x, y, z = 0, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} d\mu \int_{-\infty}^{\infty} d\sigma e^{i\sigma t} \cdot e^{i(x\mu_1 + y\mu_2)} \cdot m(\xi, \sigma, \mu) \widehat{h^*}(\mu, \xi + i\sigma).$$

Thus by the Parseval equality

$$n(\xi; \eta_t) \equiv \frac{1}{2\pi} \int_0^{+\infty} \|e^{-\xi t} \eta_t(z = 0, t)\|_{H^{-1/2}(\mathbb{R}^2)}^2 dt = \int_{\mathbb{R}^2} d\mu \int_{-\infty}^{\infty} d\sigma \frac{|m(\xi, \sigma, \mu)|^2}{(1 + |\mu|^2)^{1/2}} |\widehat{h^*}(\mu, \xi + i\sigma)|^2$$

and hence

$$n(\xi; \eta_t) \leq 4(1 + \xi^{-2})^{1/2} \int_{\mathbb{R}^2} d\mu \int_{-\infty}^{\infty} d\sigma |\widehat{h^*}(\mu, \xi + i\sigma)|^2$$

Since

$$\widehat{h^*}(\mu, \xi + i\sigma) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} dx dy \int_0^{+\infty} dt e^{-i\sigma t} \cdot e^{-i(x\mu_1 + y\mu_2)} \cdot e^{-\xi t} h^*(x, y, t), \quad \xi > 0,$$

we obtain that

$$n(\xi; \eta_t) \leq 8\pi(1 + \xi^{-2})^{1/2} \int_{\mathbb{R}^2} dx dy \int_0^{\infty} dt e^{-\xi t} |h^*(x, y, t)|^2.$$

This implies the estimate for $\|\gamma[\eta]_t\|_{L_2(0, T; H^{-1/2}(\mathbb{R}^2))}^2$ in (3.7). To obtain the corresponding bound for $\|\gamma[\eta]\|_{L_2(0, T; H^{1/2}(\mathbb{R}^2))}^2$ we use a similar argument.

The relations in (3.6) and (3.7) yield the conclusion of Lemma 3.1. \square

Remark 3.1. In Lemma 3.1 we could also take an arbitrary smooth domain \mathcal{O} instead of \mathbb{R}_+^3 . Indeed let $Q = \mathcal{O} \times (0, T)$ and $\Sigma = \partial\mathcal{O} \times (0, T)$. Assuming *a priori* $H^1(Q)$ regularity of the solution, then one can show that $\gamma[\eta_t] \in L_2(0, T; H^{-1/2}(\partial\mathcal{O}))$ for all smooth domains. The *a priori* $H^1(Q)$ regularity is automatically satisfied when the Neumann datum h^* is zero and the initial data are of finite energy ($H^1 \times L_2$). In the case when h^* is an arbitrary element of $L_2(\Sigma)$, the corresponding estimate takes the form

$$\|\gamma[\eta_t]\|_{L_2(0, T; H^{-1/2}(\mathbb{R}^2))}^2 + \|\gamma[\eta]\|_{L_2(0, T; H^{1/2}(\mathbb{R}^2))}^2 \leq C_T [\|h^*\|_{L_2(\Sigma)}^2 + \|\eta\|_{H^1(Q)}^2]. \quad (3.9)$$

The proof of this estimate can be obtained via microlocal analysis by adopting the argument given in [29, 33]. In our case when $\mathcal{O} = \mathbb{R}_+^3$ we have the estimate in (3.7) which does not contain the term $\|\eta\|_{H^1(Q)}^2$ and hence it can be extended to less regular solutions. In the case of general domains $L_2(\Sigma)$ Neumann boundary data produce in wave dynamics only $H^{2/3}(Q)$ solutions with less regular (than (3.9)) boundary traces $\gamma[\eta] \in H^{1/3}(\Sigma)$ and $\gamma[\eta_t] \in H^{-2/3}(\Sigma)$ (see [42] and also [29]). The above result is optimal and can not be improved, unless special geometries for Ω are considered. We also note that the aforementioned result improves upon [33] where the interior regularity with $L_2(\Sigma)$ Neumann data is only $\eta \in H^{1/2}(Q)$, rather than $H^{2/3}(Q)$. Thus, for general domains we observe an additional loss, with respect to (3.7), of smoothness for the boundary traces $\gamma[\eta]$ ($1/6 = 2/3 - 1/2$ of the derivative).

3.2 Linear Generation of Unperturbed Dynamics

Our main result in this section is the following assertion:

Proposition 3.3. *The operator \mathbb{A} given by (2.1) and (2.2) is skew-adjoint (i.e. $\mathbb{A}^* = -\mathbb{A}$). Thus by the Stone Theorem \mathbb{A} generates a strongly continuous isometry group $e^{\mathbb{A}t}$ in Y .*

Our calculations for the proof requires some approximation of the domain $\mathcal{D}(\mathbb{A})$ as a preliminary step.

3.2.1 Domain Approximation

We would like to build a family of approximants which allows us to justify the formal calculus occuring in the subsequent dissipativity and maximality considerations. Below we concentrate on the more difficult supersonic case $U > 1$.

Let us take arbitrary $(\phi, \psi; u, v)$ in $\mathcal{D}(\mathbb{A})$. Then

$$-U\phi_x + \psi \in H^1(\mathbb{R}_+^3), \quad -U\partial_x\psi - A(\phi + Nv) \in L_2(\mathbb{R}_+^3).$$

Since we also have that $\phi \in H^1(\mathbb{R}_+^3)$, there exist $h \in H^1(\mathbb{R}_+^3)$ and $g \in L_2(\mathbb{R}_+^3)$ (depending on the element in the domain) such that

$$-U\phi_x + \psi - r\phi = h \in H^1(\mathbb{R}_+^3), \quad -U\partial_x\psi - A(\phi + Nv) = g \in L_2(\mathbb{R}_+^3) \quad (3.10)$$

for every $r \in \mathbb{R}$. From the first relation:

$$\partial_x\psi = U\partial_x^2\phi + r\partial_x\phi + h_x.$$

Substituting into the second yields

$$-U^2\partial_x^2\phi - Ur\partial_x\phi - Uh_x - A(\phi + Nv) = g.$$

This is equivalent to

$$(U^2 - 1)\partial_x^2 \phi + Ur\phi_x - (\Delta_{y,z} + \mu)\phi = -g - Uh_x = f \in L_2(\mathbb{R}_+^3) \quad (3.11)$$

with the boundary conditions $\partial_\nu \phi = v_{ext}$, where $v \in H_0^2(\Omega)$. By the trace theorem there exists $\eta \in H^{7/2}(\mathbb{R}_+^3)$ such that $\partial_\nu \eta = v_{ext}$. Therefore we can represent

$$\phi = \phi_* + \eta,$$

where $\phi_* \in H^1(\mathbb{R}_+^3)$ is a (variational) solution to the problem

$$\begin{aligned} (U^2 - 1)\partial_x^2 \phi_* + Ur\phi_{*x} - (\Delta_{y,z} + \mu)\phi_* &= f_* \in L_2(\mathbb{R}_+^3), \\ \partial_\nu \phi_* &= 0, \end{aligned} \quad (3.12)$$

and

$$f_* = f - (U^2 - 1)\partial_x^2 \eta - Ur\eta_x + (\Delta_{y,z} + \mu)\eta.$$

Due to zero Neumann conditions the problem in (3.12) can be extended to the same equation in \mathbb{R}^3 . Therefore we can apply Fourier transform in all variables. This gives us $(x \leftrightarrow \omega, (y, z) \leftrightarrow k \in \mathbb{R}^2)$:

$$(-c_U \omega^2 + irU\omega + |k|^2 + \mu)\widehat{\phi}_* = \widehat{f}_* \in L_2,$$

where $c_U = (U^2 - 1)$. Since⁷

$$|-c_U \omega^2 + iUr\omega + |k|^2 + \mu|^2 = r^2 U^2 \omega^2 + (|k|^2 + \mu - c_U \omega^2)^2 \geq c_0(|k|^2 + \omega^2 + 1)$$

with some $c_0 > 0$ for appropriate $r = r(c_U, \mu) > 0$, the formula above leads to the solutions ϕ_* in $H^1(\mathbb{R}_+^3)$ for every $f_* \in L_2(\mathbb{R}_+^3)$. Moreover if $f_* \in H^s(\mathbb{R}_+^3)$ is such that its even (in z) extension on \mathbb{R}^3 has the same smoothness, then $\phi_* \in H^{1+s}(\mathbb{R}_+^3)$ for every $s \geq 0$; hence for $s < 3/2$ we can define a continuous affine map

$$\tau : H^s(\mathbb{R}_+^3) \rightarrow H^{s+1}(\mathbb{R}_+^3), \quad \text{where } \tau(f) = \phi,$$

where ϕ solves (3.11) with the Neumann boundary condition $\partial_\nu \phi = v_{ext} \in H^2(\mathbb{R}^2)$.

Thus, in order to find an approximate domain \mathcal{D}_n it suffices to solve (3.11) with the right hand side in $H^1(\mathbb{R}_+^3)$. Hence, we are looking for $\phi^n \in H^2(\mathbb{R}_+^3)$ such that

$$(U^2 - 1)\partial_x^2 \phi^n - \Delta_{y,z} \phi^n + \mu \phi^n + Ur\phi_x^n = -g^n - Uh_x^n \in H^1(\mathbb{R}_+^3), \quad (3.13)$$

and $\partial_\nu \phi^n = v_{ext}$ for all n , and where $h^n \rightarrow h$ in $H^1(\mathbb{R}_+^3)$ and $g^n \rightarrow g$ in $L_2(\mathbb{R}_+^3)$. Here h and g are given by (3.10).

Solving equation (3.13) with right hand side $-g^n - Uh_x^n \in H^1(\mathbb{R}_+^3)$ gives solution

$$\phi^n \in H^2(\mathbb{R}_+^3), \quad \phi_x^n \in H^1(\mathbb{R}_+^3).$$

We then define $\psi^n \equiv U\phi_x^n + r\phi^n + h^n \in H^1(\mathbb{R}_+^3)$. Then

$$U\psi_x^n + A(\phi^n + Nv) = U^2\phi_{xx}^n + Uh_x^n + A(\phi^n + Nv) + rU\phi_x^n = -g^n \rightarrow g = U\psi_x + (-\Delta + \mu)\phi$$

as desired. Additionally, since τ is affine and bounded, and each ϕ^n has the same boundary condition, we can conclude that

$$\|\phi^n - \phi\|_{H^1(\mathbb{R}_+^3)} = \|\tau(g^n + Uh_x^n) - \tau(g + Uh_x)\|_{H^1(\mathbb{R}_+^3)} \leq \|\phi^n - \phi + U(h_x^n - h_x)\|_{L_2(\mathbb{R}_+^3)} \rightarrow 0.$$

We may proceed similarly on $\psi^n = U\phi_x^n + r\phi^n + h^n$. Thus, we have obtained

⁷In the subsonic case ($c_U < 0$) the equation in (3.12) is elliptic and we can obtain better estimate.

Lemma 3.4 (Domain Approximation). *For any $y = (\phi, \psi; u, v) \in \mathcal{D}(\mathbb{A})$ there exist approximants $\phi^n \in H^2(\mathbb{R}_+^3), \psi^n \in H^1(\mathbb{R}_+^3)$ such that $y^n = (\phi^n, \psi^n; u, v) \in \mathcal{D}(\mathbb{A})$ and $y^n \rightarrow y$ in Y . Moreover*

$$\begin{aligned} U\psi_x^n + A(\phi^n + Nv) &\rightarrow U\psi_x + A(\phi + Nv), \text{ in } L_2(\mathbb{R}_+^3), \\ \psi^n - U\phi_x^n &\rightarrow \psi - U\phi_x, \text{ in } H^1(\mathbb{R}_+^3). \end{aligned}$$

As a consequence

$$(\mathbb{A}y^n, y^n)_Y \rightarrow (\mathbb{A}y, y)_Y \text{ for all } y \in \mathcal{D}(\mathbb{A}).$$

3.2.2 Dissipativity

The above approximation Lemma allows us to perform calculations on smooth functions.

Let $y^n \in \mathcal{D}(\mathbb{A})$ be the sequence of approximants as in Lemma 3.4. First, we perform the dissipativity calculation on these approximants (which allows us to move $A^{1/2}$ freely on flow terms ϕ^n and ψ^n):

$$\begin{aligned} (\mathbb{A}y^n, y^n)_Y &= \left(\begin{bmatrix} -U\partial_x\phi^n + \psi^n \\ -U\partial_x\psi^n - A(\phi^n + Nv) \\ v \\ -\mathcal{A}u + N^*A\psi^n \end{bmatrix}, \begin{bmatrix} \phi^n \\ \psi^n \\ u \\ v \end{bmatrix} \right)_{\mathcal{D}(A^{1/2}) \times L_2(\Omega) \times \mathcal{D}(\mathcal{A})^{1/2} \times L_2(\Omega)} \\ &= (A^{1/2}(\psi^n - U\partial_x\phi^n), A^{1/2}\phi^n) - (U\partial_x\psi^n + A(\phi^n + Nv), \psi^n) \\ &\quad + \langle \mathcal{A}^{1/2}v, \mathcal{A}^{1/2}u \rangle - \langle \mathcal{A}u - N^*A\psi^n, v \rangle \\ &= -U(A^{1/2}\partial_x\phi^n, A^{1/2}\phi^n) - (U\partial_x\psi^n + ANv, \psi^n) + \langle N^*A\psi^n, v \rangle. \end{aligned}$$

One can see that

$$(A^{1/2}\partial_x\phi^n, A^{1/2}\phi^n) = \int_{\mathbb{R}_+^3} \nabla\partial_x\phi^n \cdot \nabla\phi^n = \frac{1}{2} \int_{\mathbb{R}_+^3} \partial_x |\nabla\phi^n|^2 = 0$$

Similarly $(U\partial_x\psi^n, \psi^n) = 0$. Therefore

$$(\mathbb{A}y^n, y^n)_Y = -\langle ANv, \psi^n \rangle + \langle N^*A\psi^n, v \rangle = 0,$$

Furthermore, by the convergence result in Lemma 3.4, we have that for all $y \in \mathcal{D}(\mathbb{A})$

$$(\mathbb{A}y, y)_Y = \lim_{n \rightarrow \infty} (\mathbb{A}y^n, y^n)_Y = 0.$$

This gives that *both* operators \mathbb{A} and $-\mathbb{A}$ are dissipative. In other words, \mathbb{A} is skew-symmetric. This is to say

$$(\mathbb{A}y_1, y_2)_Y = -(y_1, \mathbb{A}y_2)_Y, \quad \forall y_1, y_2 \in \mathcal{D}(\mathbb{A})$$

3.2.3 Maximality

In this section we prove the maximality of the operators \mathbb{A} and $-\mathbb{A}$. For this it is sufficient to show $\mathcal{R}(\lambda - \mathbb{A}) = Y$ for every $\lambda \in \mathbb{R}, \lambda \neq 0$, i.e. for a given $F = (\phi', \psi'; u', v')$, find a $V \in \mathcal{D}(\mathbb{A})$ such that $(\lambda - \mathbb{A})V = F$. Writing this as a system, we have

$$\begin{cases} \lambda\phi + U\partial_x\phi - \psi &= \phi' \in \mathcal{D}(A^{1/2}), \\ \lambda\psi + U\partial_x\psi + A(\phi + Nv) &= \psi' \in L_2(\mathbb{R}_+^3), \\ \lambda u - v &= u' \in \mathcal{D}(\mathcal{A}^{1/2}), \\ \lambda v + \mathcal{A}u - N^*A\psi &= v' \in L_2(\Omega), \end{cases} \quad (3.14)$$

(recalling that Nv is taken to mean Nv_{ext} where v_{ext} is the extension by zero outside of Ω).

In the space Y we rewrite (3.14) in the form

$$a(V, \tilde{V}) = (F, \tilde{V})_Y, \quad (3.15)$$

where for $V = (\phi, \psi; u, v)$ and $\tilde{V} = (\tilde{\phi}, \tilde{\psi}; \tilde{u}, \tilde{v})$ we denote

$$\begin{aligned} a(V, \tilde{V}) = & (\lambda\phi + U\partial_x\phi - \psi, A\tilde{\phi})_{\mathbb{R}_+^3} \\ & + (\lambda\psi + U\partial_x\psi + A(\phi + Nv), \tilde{\psi})_{\mathbb{R}_+^3} \\ & + (\lambda u - v, \mathcal{A}\tilde{u})_\Omega + (\lambda v + \mathcal{A}u - N^*A\psi, \tilde{v})_\Omega. \end{aligned}$$

Let $\{\eta_k\} \times \{e_k\}$ be a sufficiently smooth basis in $\mathcal{D}(A^{1/2}) \times \mathcal{D}(\mathcal{A}^{1/2})$. We define an N -approximate solution to (3.15) as an element

$$V_N \in Y_N \equiv \text{Span} \{(\eta_k, \eta_l; e_m, e_n) : 1 \leq k, l, m, n \leq N\}$$

satisfying the relation

$$a(V_N, \tilde{V}) = (F, \tilde{V})_Y, \quad \forall \tilde{V} \in Y_N. \quad (3.16)$$

This can be written as a linear $4N \times 4N$ algebraic equation. Calculations on (smooth) elements V from Y_N gives

$$a(V, V) = \lambda \{ \|A^{1/2}\phi\|^2 + \|\psi\|^2 + \|\mathcal{A}^{1/2}u\|^2 + \|v\|^2 \}$$

This implies that for every $\lambda \neq 0$ the matrix which corresponds to (3.16) is non-degenerate, and therefore there exists a unique approximate solution $V_N = (\phi^N, \psi^N; u^N, v^N)$. Moreover we have that

$$a(V_N, V_N) = (F, V_N)_Y$$

which implies the a priori estimate

$$\|A^{1/2}\phi^N\|_{\mathbb{R}_+^3}^2 + \|\psi^N\|_{\mathbb{R}_+^3}^2 + \|\mathcal{A}^{1/2}u^N\|_\Omega^2 + \|v^N\|_\Omega^2 \leq \frac{1}{\lambda^2} \|F\|_Y^2$$

Thus $\{V_N\}$ is weakly compact in Y . This allows us to make limit transition in (3.16) to obtain the equality

$$a(V, \tilde{V}) = (F, \tilde{V})_Y, \quad \forall \tilde{V} \in Y_M, \quad \forall M,$$

for some $V \in Y$. Thus (3.14) is satisfied in the sense distributions. This proves maximality of both operators \mathbb{A} and $-\mathbb{A}$.

Since both operators \mathbb{A} and $-\mathbb{A}$ are maximal and dissipative, we can apply [12, Corollary 2.4.11. p.25] and conclude that the operator \mathbb{A} is skew-adjoint with respect to Y . (This later conclusion also follows from the fact that the defect indices of \mathbb{A} are zero, [43]). This completes the proof of Proposition 3.3.

The fact that \mathbb{A} is skew-adjoint simplifies calculations later in the treatment. In what follows, we use $\mathcal{D}(\mathbb{A})$ and $\mathcal{D}(\mathbb{A}^*)$ interchangeably.

Remark 3.2. To conclude this section, we mention that for $y_0 \in Y$ the C_0 -group $e^{\mathbb{A}t}$ generates a mild solution $y(t) = e^{\mathbb{A}t}y_0$ to the PDE problem given in (1.15) without the boxed term (see also (3.17) below with $k = 0$). We also note that adding a linearly bounded perturbation to the dynamics will not affect the generation of a C_0 group. In particular, the addition of internal

damping for the plate of the form ku_t , $k > 0$ on the LHS of (1.1) does not affect generation and we can construct C_0 -group $T_k(t)$ which corresponds to the problem

$$\begin{cases} (\partial_t + U\partial_x)\phi = \psi & \text{in } \mathbb{R}_+^3 \times (0, T), \\ (\partial_t + U\partial_x)\psi = \Delta\phi - \mu\phi & \text{in } \mathbb{R}_+^3 \times (0, T), \\ \partial_\nu\phi = -u_t \cdot \mathbf{1}_\Omega(\mathbf{x}) & \text{on } \mathbb{R}_{\{(x,y)\}}^2 \times (0, T), \\ u_{tt} + ku_t + \Delta^2 u = \gamma[\psi] & \text{in } \Omega \times (0, T), \\ u = \partial_\nu u = 0 & \text{in } \partial\Omega \times (0, T). \end{cases} \quad (3.17)$$

Moreover, this damping term ku_t does not alter $\mathcal{D}(\mathbb{A})$ or $\mathcal{D}(\mathbb{A}^*)$, and hence the analysis to follow concerning the full dynamics ($\mathbb{A} + \mathbb{P}$, and the addition of nonlinearity) is valid presence of interior damping. We plan use this observation in future studies of this model.

3.3 Variation of Parameters and Perturbed Linear Dynamics

We begin with some preparations.

3.3.1 Preliminaries

We would like to introduce a perturbation to the operator \mathbb{A} which will produce the non-monotone flow-structure problem above. For this we define an operator $\mathbb{P} : Y \rightarrow \mathcal{R}(\mathbb{P})$ as follows:

$$\mathbb{P} \begin{pmatrix} \phi \\ \psi \\ u \\ v \end{pmatrix} = \mathbb{P}_\# [u] \equiv \begin{pmatrix} 0 \\ -UAN\partial_x u \\ 0 \\ 0 \end{pmatrix} \quad (3.18)$$

Specifically, the problem in (1.15) has the abstract Cauchy formulation:

$$y_t = (\mathbb{A} + \mathbb{P})y, \quad y(0) = y_0,$$

where $y_0 \in Y$ will produce semigroup (mild) solutions to the corresponding integral equation, and $y_0 \in \mathcal{D}(\mathbb{A})$ will produce classical solutions. To find solutions to this problem, we will consider a fixed point argument, which necessitates interpreting and solving the following inhomogeneous problem, and then producing the corresponding estimate on the solution:

$$y_t = \mathbb{A}y + \mathbb{P}_\# \bar{u}, \quad t > 0, \quad y(0) = y_0, \quad (3.19)$$

for a given \bar{u} . To do so, we must understand how \mathbb{P} acts on Y (and thus $\mathbb{P}_\#$ on $H_0^2(\Omega)$).

To motivate the following discussion, consider for $y \in Y$ and $z = (\phi, \bar{\psi}; \bar{u}, \bar{v})$ the formal calculus (with Y as the pivot space)

$$(\mathbb{P}y, z)_Y = (\mathbb{P}_\# [u], z)_Y = -U(AN\partial_x u, \bar{\psi}) = -U \langle \partial_x u, \gamma[\bar{\psi}] \rangle. \quad (3.20)$$

Hence, interpreting the operator \mathbb{P} (via duality) is contingent upon the ability to make sense of $\gamma[\bar{\psi}]$, which *can* be done if $\gamma[\bar{\psi}] \in H^{-1/2}(\Omega)$. In what follows, we show that the trace estimate on ψ for mild solutions of (3.19) allows us to justify the formal energy method (multiplication of (3.19) by the solution y) in order to perform a fixed point argument.

To truly get to the heart of this matter, we must interpret the following variation of parameters statement for $\bar{u} \in C(\mathbb{R}_+; H_0^2(\Omega))$ (which will ultimately be the solution to (3.19)):

$$y(t) = e^{\mathbb{A}t} y_0 + \int_0^t e^{\mathbb{A}(t-s)} \mathbb{P}_\# [\bar{u}(s)] ds. \quad (3.21)$$

To do so, we make use of the work in [30] and write (with some $\lambda \in \mathbb{R}$, $\lambda \neq 0$):

$$y(t) = e^{\mathbb{A}t}y_0 + (\lambda - \mathbb{A}) \int_0^t e^{\mathbb{A}(t-s)}(\lambda - \mathbb{A})^{-1} \mathbb{P}_{\#}[\bar{u}(s)] ds, \quad (3.22)$$

initially interpreting this solution as an element of $[\mathcal{D}(\mathbb{A}^*)]' = [\mathcal{D}(\mathbb{A})]'$, i.e., by considering the solution $y(t)$ in (3.22) above acting on an element of $\mathcal{D}(\mathbb{A}^*)$.

3.3.2 Abstract Semigroup Convolution

At this point we cast the discussion of the perturbation \mathbb{P} (acting outside of Y) into the context of abstract boundary control. By doing this we simplify and distill our exposition, and moreover, provide a context for further boundary control considerations. For this discussion, we select and cite some results from [30, pp.645-653] which will be used in this section.

Let X and \mathcal{U} be reflexive Banach spaces. We assume that

- (C1) A is a linear operator which generates a strongly continuous semigroup e^{At} on X .
- (C2) B is a linear continuous operator from \mathcal{U} to $[\mathcal{D}(A^*)]'$ (duality with respect to the pivot space X), or equivalently, $(\lambda - A)^{-1}B \in \mathcal{L}(\mathcal{U}, X)$ for all $\lambda \in \rho(A)$.

For fixed $0 < T < \infty$ and $u \in L_1(0, T; \mathcal{U})$ we define the convolution operator

$$(Lu)(t) \equiv \int_0^t e^{A(t-s)} Bu(s) ds, \quad 0 \leq t \leq T,$$

corresponding to the mild solution

$$x(t) = e^{At}x_0 + (Lu)(t), \quad 0 \leq t \leq T,$$

of the abstract inhomogeneous equation

$$x_t = Ax + Bu \in [\mathcal{D}(A^*)]', \quad x(0) = x_0,$$

with the input function $Bu(t)$.

Theorem 3.5 (Inhomogeneous Abstract Equations). *Let X and \mathcal{U} be reflexive Banach spaces and the conditions in (C1) and (C2) be in force. Then*

1. *The semigroup e^{At} can be extended to the space $[\mathcal{D}(A^*)]'$.*
2. *L is continuous from $L_p(0, T; X)$ to $C(0, T; [\mathcal{D}(A^*)]')$ for every $p \in [1, \infty]$.*
3. *If $u \in C^1(0, T; X)$, then $Lu \in C(0, T; X)$.*
4. *The condition*

(C3) *There exists a constant $C_T > 0$ such that*

$$\int_0^T \|B^* e^{A^*t} x^*\|_{\mathcal{U}^*}^2 dt \leq C_T \|x^*\|_{X^*}^2, \quad \forall x^* \in \mathcal{D}(A^*) \subset X^*,$$

is equivalent to the regularity property

$$L : L_2(0, T; \mathcal{U}) \rightarrow C(0, T; X) \text{ is continuous,}$$

i.e., there exists a constant $k_T > 0$ such that

$$\|Lu\|_{C(0, T; X)} \leq k_T \|u\|_{L_2(0, T; \mathcal{U})}. \quad (3.23)$$

5. Lastly, assume additionally that A generates a strongly continuous group e^{At} (e.g., if A is skew-adjoint) and suppose that $L : L_2(0, T; \mathcal{U}) \rightarrow L_2(0, T; X)$ is continuous. Then (C3) is satisfied and thus we have the estimate in (3.23) in this case.

3.3.3 Application of the Abstract Scheme

We now introduce the auxiliary space which will be needed in the proof of the next lemma:

$$Z \equiv \left\{ y = \begin{pmatrix} \phi \\ \psi \\ u \\ v \end{pmatrix} \in Y : -U\partial_x\phi + \psi \in H^1(\mathbb{R}_+^3) \right\}$$

endowed with the norm

$$\|y\|_Z = \|y\|_Y + \|-U\partial_x\phi + \psi\|_{H^1(\mathbb{R}_+^3)}.$$

One can see that Z is dense in Y . We also note that by Lemma 2.1, $\mathcal{D}(\mathbb{A}) \subset Z$ and thus $Z' \subset [\mathcal{D}(\mathbb{A})]'$ with continuous embedding.

Lemma 3.6. *The operator $\mathbb{P}_\#$ given by (3.18) is a bounded linear mapping from $H_0^2(\Omega)$ into Z' . Moreover, the following estimates are in force:*

$$\|\mathbb{P}_\#[u]\|_{Z'} \leq C_U \|u\|_{H^2(\Omega)}, \quad \forall u \in H_0^2(\Omega), \quad (3.24)$$

and also (with $\lambda \in \mathbb{R}$, $\lambda \neq 0$):

$$\|(\lambda - \mathbb{A})^{-1} \mathbb{P}_\#[u]\|_Y \leq C_{U,\lambda} \|u\|_{H^2(\Omega)}, \quad \forall u \in H_0^2(\Omega). \quad (3.25)$$

In the latter case we understand $(\lambda - \mathbb{A})^{-1} : [\mathcal{D}(\mathbb{A})]' \mapsto Y$ as the inverse to the operator $\lambda - \mathbb{A}$ which is extended to a mapping from Y to $[\mathcal{D}(\mathbb{A})]'$. We also have that (3.24) and (3.25) imply that \mathbb{P} maps Y into Z' and

$$\|\mathbb{P}y\|_{Z'} \leq C \|y\|_Y \quad \text{and} \quad \|(\lambda - \mathbb{A})^{-1} \mathbb{P}y\|_Y \leq C \|y\|_Y, \quad \forall y \in Y.$$

Proof. For $u \in H_0^2(\Omega)$ and $\bar{y} = (\bar{\phi}, \bar{\psi}; \bar{u}, \bar{v}) \in Z$, from (3.20) we have

$$|(\mathbb{P}_\#[u], \bar{y})_Y| = U |(AN\partial_x u, \bar{\psi})| = U |\langle \partial_x u, \gamma[\bar{\psi}] \rangle|.$$

Since $\gamma[\bar{\psi}] = \gamma[-U\partial_x\bar{\phi} + \bar{\psi}] + U\partial_x\gamma[\bar{\phi}]$, we have from the trace theorem that

$$\|\gamma[\bar{\psi}]\|_{H^{-1/2}(\mathbb{R}^2)} \leq C \left[\|(-U\partial_x\bar{\phi} + \bar{\psi})\|_{H^1(\mathbb{R}_+^3)} + \|\bar{\phi}\|_{H^1(\mathbb{R}_+^3)} \right] \leq C \|\bar{y}\|_Z.$$

Therefore

$$|(\mathbb{P}_\#[u], \bar{y})_Y| \leq C(U) \|\partial_x u\|_{H^{1/2}(\Omega)} \|\gamma[\bar{\psi}]\|_{H^{-1/2}(\mathbb{R}^2)} \leq C(U) \|u\|_{H^2(\Omega)} \|\bar{y}\|_Z.$$

Thus

$$\|\mathbb{P}_\#[u]\|_{Z'} = \sup \{ |(\mathbb{P}_\#[u], \bar{y})_Y| : \bar{y} \in Z, \|\bar{y}\|_Z = 1 \} \leq C(U) \|u\|_{H^2(\Omega)},$$

which implies (3.24). The relation in (3.25) follows from (3.24) and the boundedness of the operator $(\lambda - \mathbb{A})^{-1} : [\mathcal{D}(\mathbb{A})]' \mapsto Y$. \square

We may now consider mild solutions to the problem given in (3.19). Applying general results on C_0 -semigroups (see [35]) we arrive at the following assertion.

Proposition 3.7. *Let $\bar{u} \in C^1([0, T]; H_0^2(\Omega))$ and $y_0 \in Y$. Then $y(t)$ given by (3.21) belongs to $C([0, T]; Y)$ and is a strong solution to (3.19) in $[\mathcal{D}(\mathbb{A})]'$, i.e. in addition we have that*

$$y \in C^1((0, T); [\mathcal{D}(\mathbb{A})]')$$

and (3.19) holds in $[\mathcal{D}(\mathbb{A})]'$ for each $t \in (0, T)$.

Proof. The result follows from the integration by parts formula

$$\begin{aligned} (L\bar{u})(t) &\equiv \int_0^t e^{A(t-s)} \mathbb{P}_\# [\bar{u}(s)] ds = (\lambda - A) \int_0^t e^{A(t-s)} (\lambda - A)^{-1} \mathbb{P}_\# [\bar{u}(s)] ds \\ &= \lambda \int_0^t e^{A(t-s)} (\lambda - A)^{-1} \mathbb{P}_\# [\bar{u}(s)] ds - \int_0^t e^{A(t-s)} (\lambda - A)^{-1} \mathbb{P}_\# \left[\frac{d}{ds} \bar{u}(s) \right] ds \\ &\quad - e^{At} (\lambda - A)^{-1} \mathbb{P}_\# [\bar{u}(0)] + (\lambda - A)^{-1} \mathbb{P}_\# [\bar{u}(t)] \\ &\in C(0, T; X), \end{aligned}$$

and by applying [35, Corollary 2.5, p.107] to show that y is a strong solution in $[\mathcal{D}(\mathbb{A})]'$. \square

Proposition 3.7 implies that $y(t)$ satisfies the variational relation

$$\partial_t(y(t), h)_Y = -(y(t), \mathbb{A}h)_Y + (\mathbb{P}_\# [\bar{u}(t)], h)_Y, \quad \forall h \in \mathcal{D}(\mathbb{A}). \quad (3.26)$$

In our context in application of Theorem 3.5 we have that $X = Y$, where Y is Hilbert space given by (1.12), $\mathcal{U} = H_0^2(\Omega)$ and $B = \mathbb{P}_\#$ is defined by (3.18) as an operator from $\mathcal{U} = H_0^2(\Omega)$ into Z' (see Lemma 3.6). One can see from (3.20) that the adjoint operator $\mathbb{P}_\#^* : Z \mapsto H^{-2}(\Omega)$ is given

$$\mathbb{P}_\#^* z = U [\partial_x N^* A \bar{\psi}]|_\Omega = U \partial_x \gamma[\bar{\psi}]|_\Omega, \quad \text{for } z = \begin{pmatrix} \bar{\phi} \\ \bar{\psi} \\ \bar{u} \\ \bar{v} \end{pmatrix} \in Z.$$

Condition (C3) in Theorem 3.5(4) is then paraphrased by writing

$$y \mapsto \mathbb{P}_\#^* e^{\mathbb{A}^*(T-t)} y \equiv \mathbb{P}_\#^* e^{\mathbb{A}t} y_T : \text{continuous } Y \rightarrow L_2(0, T; H^{-2}(\Omega)),$$

where $y_T = e^{\mathbb{A}^*T} y = e^{-\mathbb{A}T} y$ (we use here the fact that \mathbb{A} is a skew-adjoint generator).

Let us denote by $e^{\mathbb{A}t} y_T \equiv w_T(t) = (\phi(t), \psi(t); u(t), v(t))$ the solution of the linear problem in (2.5) with initial data y_T . Then from the trace estimate in (3.4) we have that

$$\begin{aligned} \|\mathbb{P}_\#^* e^{\mathbb{A}t} y_T\|_{L_2(0, T; H^{-2}(\Omega))}^2 &= U \int_0^T \|\partial_x \gamma[\psi(t)]\|_{H^{-2}(\Omega)}^2 dt \\ &\leq C \int_0^T \|\gamma[\psi(t)]\|_{H^{-1/2}(\Omega)}^2 dt \leq C_T \left(E_{fl}(0) + \int_0^T \|v(t)\|_{L_2(\Omega)}^2 dt \right) \\ &\leq C_T \|y_T\|_Y = C_T \|e^{-\mathbb{A}T} y\|_Y = C_T \|y\|_Y. \end{aligned}$$

In the last equality we also use that $e^{\mathbb{A}t}$ is a C_0 -group of isometries.

Now we are fully in a position to use Theorem 3.5 which leads to the following assertion.

Theorem 3.8 (L Regularity). *Let $T > 0$ be fixed, $y_0 \in Y$ and $\bar{u} \in C([0, T]; H_0^2(\Omega))$. Then the mild solution*

$$y(t) = e^{\mathbb{A}t} y_0 + L[\bar{u}](t) \equiv e^{\mathbb{A}t} y_0 + \int_0^t e^{\mathbb{A}(t-s)} \mathbb{P}_\# [\bar{u}(s)] ds$$

to problem (3.19) in $[\mathcal{D}(\mathbb{A})]'$ belongs to the class $C([0, T]; Y)$ and enjoys the estimate

$$\max_{\tau \in [0, t]} \|y(\tau)\|_Y \leq \|y_0\|_Y + k_T \|\bar{u}\|_{L_2(0, t; H_0^2(\Omega))}, \quad \forall t \in [0, T]. \quad (3.27)$$

Remark 3.3. The discussion beginning at Theorem 3.5 and ending with the estimate in (3.27) demonstrates how the perturbation \mathbb{P} acting outside of Y is regularized when incorporated into the operator L ; namely, the variation of parameters operator L is a priori only continuous from $L_2(0, T; \mathcal{U})$ to $C(0, T; [\mathcal{D}(\mathbb{A}^*)]')$. However, we have shown that the additional “hidden” regularity of the trace of ψ for solutions to (1.7) allows us to bootstrap L to be continuous from $L_2(0, T; \mathcal{U})$ to $C(0, T; Y)$ (with corresponding estimate) via the abstract Theorem 3.5. This result essentially justifies *formal* energy methods on the equation (3.19) in order to set up a fixed point argument (which will follow in the next section).

For completeness, we also include a direct proof of an estimate of the type (3.27) in the Appendix, independent of the abstract boundary control framework presented in Theorem 3.5.

3.4 Construction of a Generator

Let $\mathbb{X}_t = C((0, t]; Y)$. Now, take $\bar{y} = (\bar{\phi}, \bar{\psi}; \bar{u}, \bar{v}) \in \mathbb{X}_t$ and $y_0 \in Y$, and introduce the map $\mathcal{F} : \bar{y} \rightarrow y$ given by

$$y(t) = e^{\mathbb{A}t} y_0 + L[\bar{u}](t),$$

i.e. y solves

$$y_t = \mathbb{A}y + \mathbb{P}_\# \bar{u}, \quad y(0) = y_0,$$

in the generalized sense, where $\mathbb{P}_\#$ is defined in (3.18). It follows from (3.27) that for $\bar{y}_1, \bar{y}_2 \in \mathbb{X}_t$

$$\begin{aligned} \|\mathcal{F}\bar{y}_1 - \mathcal{F}\bar{y}_2\|_{\mathbb{X}_t} &\leq k_T \|\bar{u}_1 - \bar{u}_2\|_{L_2(0, t; H_0^2(\Omega))} \\ &\leq k_T \sqrt{t} \max_{\tau \in [0, t]} \|\bar{u}_1 - \bar{u}_2\|_{H^2(\Omega)} \leq k_T \sqrt{t} \|\bar{y}_1 - \bar{y}_2\|_{\mathbb{X}_t}. \end{aligned}$$

Hence there is $0 < t_* < T$ and $q < 1$ such that

$$\|\mathcal{F}\bar{y}_1 - \mathcal{F}\bar{y}_2\|_{\mathbb{X}_t} \leq q \|\bar{y}_1 - \bar{y}_2\|_{\mathbb{X}_t}$$

for every $t \in (0, t_*]$. This implies that on the interval $[0, t_*]$ the problem

$$y_t = \mathbb{A}y + \mathbb{P}y, \quad t > 0, \quad y(0) = y_0,$$

has a local in time unique (mild) solution defined now in Y . This above local solution can be extended to a global solution in finitely many steps by linearity. Thus there exists a unique function $y = (\phi, \psi; u, v) \in C(\mathbb{R}_+; Y)$ such that

$$y(t) = e^{\mathbb{A}t} y_0 + \int_0^t e^{\mathbb{A}(t-s)} \mathbb{P}[y(s)] ds \quad \text{in } Y \text{ for all } t > 0. \quad (3.28)$$

It also follows from the analysis above that

$$\|y(t)\|_Y \leq C_T \|y_0\|_Y, \quad t \in [0, T], \quad \forall T > 0.$$

Thus the problem (3.28) generates strongly continuous semigroup $\hat{T}(t)$ in Y . Additionally, due to (3.26) we have

$$(y(t), h)_Y = (y_0, h)_Y + \int_0^t [-(y(\tau), \mathbb{A}h)_Y + (\mathbb{P}[y(\tau)], h)_Y] d\tau, \quad \forall h \in \mathcal{D}(\mathbb{A}), \quad t > 0.$$

Using the same idea as in subsonic case [17] (which relies on Ball's Theorem [4] and ideas presented in [19]), we can conclude that the generator $\hat{\mathbb{A}}$ of $\hat{T}(t)$ has the form

$$\hat{\mathbb{A}}z = \mathbb{A}z + \mathbb{P}z, \quad z \in \mathcal{D}(\hat{\mathbb{A}}) = \{z \in Y : \mathbb{A}z + \mathbb{P}z \in Y\}$$

(we note that the sum $\mathbb{A}z + \mathbb{P}z$ is well-defined as an element in $[\mathcal{D}(\mathbb{A})]'$ for every $z \in Y$). Hence, the semigroup $e^{\hat{\mathbb{A}}t}y_0$ is a generalized solution for $y_0 \in Y$ (resp. a classical solution for $y_0 \in \mathcal{D}(\hat{\mathbb{A}})$) to (1.15) on $[0, T]$ for all $T > 0$.

$$\mathcal{D}(\mathbb{A} + \mathbb{P}) \equiv \left\{ y \in Y \left| \begin{array}{l} -U\partial_x\phi + \psi \in H^1(\mathbb{R}_+^3), \\ -U\partial_x\psi - A(\phi + N(v + U\partial_x u)) \in L_2(\mathbb{R}_+^3) \\ v \in \mathcal{D}(\mathcal{A}^{1/2}) = H_0^2(\Omega), \quad -\mathcal{A}u + N^*A\psi \in L_2(\Omega) \end{array} \right. \right\} \quad (3.29)$$

Now we can conclude the proof of Theorem 1.1 in the same way as in [17] by considering bounded perturbation of generator of the term

$$C(y) = (0, \mu\phi; 0, 0).$$

Indeed, the function $y(t)$ is a generalized solution corresponding to the generator $\mathbb{A} + \mathbb{P}$ with the domain defined in (3.29). This proves the first statement in Theorem 1.1. As for the second statement (regularity), the invariance of the domain $\mathcal{D}(\mathbb{A} + \mathbb{P})$ under the flow implies that solutions originating in Y_1 will remain in $C([0, T]; Y_1)$. After identifying $\phi_t = \psi - U\partial_x\phi$ one translates the membership in the domain into membership in Y_1 . Elliptic regularity applied to biharmonic operator yields the precise regularity results defining *strong* solutions. The proof of Theorem 1.1 is thus completed.

3.5 Nonlinear Semigroup and Completion of the Proof of Theorem 1.2

To prove Theorem 1.2 we also use the same idea as in [17]. As an intermediate step we obtain the following theorem.

Theorem 3.9. *Let $\mathcal{F}(y) = (0, F_1(\phi); 0, F_2(u))$ where*

$$F_1 : H^1(\mathbb{R}_+^3) \rightarrow L_2(\mathbb{R}_+^3) \quad \text{and} \quad F_2 : H_0^2(\Omega) \rightarrow L_2(\Omega) \quad \text{are locally Lipschitz}$$

in the sense that every $R > 0$ there exists $c_R > 0$ such that

$$\|F_1(\phi) - F_1(\phi^*)\|_{\mathbb{R}_+^3} \leq c_R \|\phi - \phi^*\|_{1, \mathbb{R}_+^3} \quad \text{and} \quad \|F_2(u) - F_2(u^*)\|_{\Omega} \leq c_R \|u - u^*\|_{2, \Omega}$$

for all $\phi, \phi^ \in H^1(\mathbb{R}_+^3)$ and $u, u^* \in H_0^2(\Omega)$ such that $\|\phi\|_{1, \mathbb{R}_+^3}, \|\phi^*\|_{1, \mathbb{R}_+^3}, \|u\|_{2, \Omega}, \|u^*\|_{2, \Omega} \leq R$. Then the equation*

$$y_t = (\mathbb{A} + \mathbb{P})y + \mathcal{F}(y), \quad y(0) = y_0 \in Y \quad (3.30)$$

has a unique local-in-time generalized solution $y(t)$ (which is also weak). Moreover, for $y_0 \in \mathcal{D}(\mathbb{A} + \mathbb{P})$, the corresponding solution is strong.

In both cases, when $t_{\max}(y_0) < \infty$, we have that $\|y(t)\|_Y \rightarrow \infty$ as $t \nearrow t_{\max}(y_0)$.

Proof. This is a direct application of Theorem 1.4 [35, p.185] and localized version of Theorem 1.6 [35, p.189]. \square

In order to guarantee global solutions, one must have more information on the nature of the nonlinear term. The following result provides relevant abstract conditions imposed on nonlinear terms which can be verified:

Theorem 3.10. *We assume that f is locally Lipschitz from $H_0^2(\Omega)$ into $L_2(\Omega)$ and there exists C^1 -functional $\Pi(u)$ on $H_0^2(\Omega)$ such that f is a Fréchet derivative of $\Pi(u)$, $f(u) = \Pi'(u)$. Moreover we assume that $\Pi(u)$ is locally bounded on $H_0^2(\Omega)$, and there exist $\eta < 1/2$ and $C \geq 0$ such that*

$$\eta \|\Delta u\|_\Omega^2 + \Pi(u) + C \geq 0, \quad \forall u \in H_0^2(\Omega). \quad (3.31)$$

Then the generalized solution in (3.30) is global, i.e. $t_{max} = \infty$.

Proof. The relation in (3.31) implies that the full energy $\mathcal{E}(t)$ defined in (1.8) admits the estimates

$$\mathcal{E}(t) \geq c_0 \left(\|\psi\|_{\mathbb{R}_+^2}^2 + \|\phi\|_{\mathbb{R}_+^2}^2 + \|u_t\|_\Omega^2 + \|\Delta u\|_\Omega^2 \right) - c_1 \equiv c_0 \|y(t)\|_Y^2 - c_1$$

for some positive c_i . Therefore using the energy relation in (1.9) on the existence time interval and the flow trace regularity (see Lemma 3.1) we can conclude that

$$\begin{aligned} c_0 \|y(t)\|_Y^2 &\leq c_1 + \mathcal{E}(0) + \int_0^t \|\Delta u(\tau)\|_\Omega \|\psi(\tau)\|_{H^{-1/2}(\mathbb{R}_+^2)} d\tau \\ &\leq C(y_0) + C_T \int_0^t \|y(\tau)\|_Y^2 d\tau. \end{aligned}$$

Thus the solution $y(t)$ cannot blow up at finite time, i.e., $t_{max} = \infty$. \square

Thus, in order to complete the proof of the first part of Theorem 1.2 one needs to verify that the nonlinear forcing $f(u)$ given in Assumption 1.1 comply with the requirements of Theorem 3.10.

3.5.1 Verification of the Hypotheses

We note the examples of forcing terms described above satisfy conditions of Theorem 3.10.

Step 1: Kirchhoff model. Indeed, in the case of the *Kirchhoff model*, the embeddings $H^s(\Omega) \subset L_\infty(\Omega)$ for $s > 3/2$ implies that

$$\|f(u_1) - f(u_2)\|_\Omega \leq c_R \|u_1 - u_2\|_{H^{2-\delta}(\Omega)} \quad (3.32)$$

for every $u_i \in H_0^2(\Omega)$ with $\|u\|_{H^2(\Omega)} \leq R$. The functional $\Pi(u)$ has the form

$$\Pi(u) = \int_\Omega F(u(x)) dx \quad \text{with} \quad F(s) = \int_0^s f(\xi) d\xi$$

It follows from (1.2) that there exist $\gamma < \lambda_1$ and $C \geq 0$ such that $F(s) \geq -\gamma s^2/2 - C$ for all $s \in \mathbb{R}$. This implies (3.31).

Step 2: Von Karman model. In the case of the *von Karman model* the arguments are more subtle. We rely on sharp regularity of Airy stress function [23] and also Corollary 1.4.5 in [16].

$$\|\Delta^{-2}[u, w]\|_{W^{2,\infty}(\Omega)} \leq C \|u\|_{2,\Omega} \|w\|_{2,\Omega}$$

where Δ^2 denotes biharmonic operator with zero clamped boundary conditions. The above yields

$$\|v(u)\|_{W^{2,\infty}(\Omega)} \leq C \|u\|_{2,\Omega}^2$$

which in turn implies that the Airy stress function $v(u)$ defined in (1.3) satisfies the inequality

$$\|[u_1, v(u_1)] - [u_2, v(u_2)]\|_\Omega \leq C (\|u_1\|_{2,\Omega}^2 + \|u\|_{2,\Omega} \|u_2\|_{2,\Omega}^2) \|u_1 - u_2\|_{2,\Omega} \quad (3.33)$$

(see Corollary 1.4.5 in [16]). Thus, $f(u) = -[u, v(u) + F_0]$ is locally Lipschitz on $H_0^2(\Omega)$.

The potential energy Π has the form

$$\Pi(u) = \frac{1}{4} \int_{\Omega} [|v(u)|^2 - 2([u, F_0])u] dx$$

and possesses the property in (3.31), see, e.g., Lemma 1.5.4 [16, Chapter 1]. It is worthwhile to note that the property (3.31) is related to the validity of the maximum principle for Monge Ampere equations. For functions $u \in H^2(\Omega)$ one has

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \frac{\text{diam}\Omega}{\sqrt{\pi}} \| [u, u] \|_{L_1(\Omega)}^{1/2}$$

Thus for $u \in H^2(\Omega) \cap H_0^1(\Omega)$ we have (see Lemma 1.5.5 in [16]):

$$\max_{\Omega} |u(x)| \leq \frac{\text{diam}\Omega}{\sqrt{\pi}} \| [u, u] \|_{L_1(\Omega)}^{1/2}$$

The above uniqueness property is critical in proving (3.31) for any function $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Step 3: Berger's model. One can also see that the Berger model satisfies (3.32) with $\delta = 0$ and (3.31) holds; for details see [13, Chapter 4] and [15, Chapter 7].

3.5.2 Completion of the Proof of Theorem 1.2

For the proof of Theorem 1.2 it suffices to apply Theorem 3.10 along with the estimates stated above. These estimates assert that the hypotheses of Theorem 3.10 have been verified for all three nonlinear models under consideration.

Concerning strong solutions we notice first that on the strength of the estimate (3.33) the domain of $(\mathbb{A} + \mathbb{P}) + \mathcal{F}$ in the von Karman case is the same as the domain $\mathcal{D}(\mathbb{A} + \mathbb{P})$. The same holds for the other two models. The local Lipschitz property of the nonlinear terms, along with global bounds on solutions, allows us to claim the invariance of the domain of the nonlinear flow. Thus for the initial data in Y_1 one has that the solution $y = (\phi, \phi_t; u, u_t) \in L_{\infty}(0, T; Y_1)$ (as in the argument of Theorem 3.9 we refer to Theorem 1.6 [35, p.189]). This implies

$$\begin{aligned} y &\in C([0, T]; Y), \quad \phi_t \in L_{\infty}(0, T; H^1(\mathbb{R}_+^3)), \quad u_t \in L_{\infty}(0, T; H_0^2(\Omega)), \\ \phi_{tt} &= -2\phi_{xt} - U^2\phi_{xx} + \Delta\phi \in L_{\infty}(0, T; L_2(\mathbb{R}_+^3)), \\ u_{tt} &= -\Delta^2 u + \gamma[\phi_t + U\phi_x] - f(u) = -\Delta^2 u + U\gamma[\phi_x] + \gamma[\phi_t] - f(u) \in L_{\infty}(0, T; L_2(\Omega)), \\ \text{the above implies via elliptic theory and Sobolev's embeddings} \\ \Delta^2 u &\in L_{\infty}(0, T; H^{-1/2}(\Omega)) \rightarrow u \in L_{\infty}(0, T; H^{7/2}(\Omega)), \\ (U^2 - 1)\phi_{xx} - \phi_{zz} - \phi_{yy} &\in L_{\infty}(0, T; L_2(\mathbb{R}_+^3)), \quad \phi_z|_{z=0} = u_t + Uu_x \in L_{\infty}(0, T; L_2(\mathbb{R}^2)). \end{aligned}$$

The above relations imply the regularity properties required from strong solutions.

The regularity postulated for strong solutions is sufficient in order to define variational forms describing the solutions. The existence and uniqueness of weak solutions follow by viewing generalized solutions as the strong limits of strong solutions. This, along with Lipschitz estimates satisfied by nonlinear forces, allows a passage with the limit on strong solutions. This completes the proof of Theorem 1.2.

In conclusion we note that the well-posedness results presented in this treatment are a necessary first step in studying long-time behavior of solutions. This can be done without the addition of damping mechanisms (see [14] and [16, Remark 12.4.8]) or in the presence of control

theoretic damping, e.g. boundary or interior dissipation (see [16, 32]. In either case, the next step will be to show the existence of global attracting sets for the plate component of the model, and analyze their properties (i.e., compactness, dimensionality, and regularity).

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5 Appendix

5.1 Direct Proof of Estimate (3.27) for Fixed Point Statement

Let $\bar{u} \in C^2([0, T]; H_0^2(\Omega))$ and let $y(t) = (\phi(t), \psi(t); u(t), v(t)) \in C([0, T]; Y)$ be a mild solution to (3.19). This implies that $y(t)$ is a (distributional) solution to problem

$$\begin{cases} (\partial_t + U\partial_x)\phi = \psi & \text{in } \mathbb{R}_+^3 \times (0, T), \\ (\partial_t + U\partial_x)\psi = \Delta\phi - \mu\phi & \text{in } \mathbb{R}_+^3 \times (0, T), \\ \partial_\nu\phi = -(\partial_t u + U\partial_x w) \cdot \mathbf{1}_\Omega(\mathbf{x}) & \text{on } \mathbb{R}_{\{(x,y)\}}^2 \times (0, T), \\ u_{tt} + \Delta^2 u = \gamma[\psi] & \text{in } \Omega \times (0, T), \\ u = \partial_\nu u = 0 & \text{in } \partial\Omega \times (0, T). \end{cases} \quad (5.1)$$

It follows from the trace theorem that there exists η from the class $C^2([0, T]; H^2(\mathbb{R}_+^3))$ such that

$$\partial_\nu \eta = -U[\partial_x \bar{u}]_{ext} \quad \text{on } \mathbb{R}_{\{(x,y)\}}^2 \times (0, T).$$

Let $\tilde{\phi} = \phi - \eta$. Then it follows from (5.1) that $\tilde{y}(t) = (\tilde{\phi}(t), \psi(t); u(t), v(t)) \in C([0, T]; Y)$ solves (inhomogeneous) problem

$$\begin{cases} (\partial_t + U\partial_x)\tilde{\phi} = \psi + f_1 & \text{in } \mathbb{R}_+^3 \times (0, T), \\ (\partial_t + U\partial_x)\psi = \Delta\tilde{\phi} - \mu\tilde{\phi} + f_2 & \text{in } \mathbb{R}_+^3 \times (0, T), \\ \partial_\nu \tilde{\phi} = -\partial_t u \cdot \mathbf{1}_\Omega(\mathbf{x}) & \text{on } \mathbb{R}_{\{(x,y)\}}^2 \times (0, T), \\ u_{tt} + \Delta^2 u = \gamma[\psi] & \text{in } \Omega \times (0, T), \\ u = \partial_\nu u = 0 & \text{in } \partial\Omega \times (0, T). \end{cases} \quad (5.2)$$

where

$$f_1 = -(\partial_t + U\partial_x)\eta, \quad f_2 = (\Delta - \mu)\eta.$$

Problem (5.2) can be written in the form

$$\tilde{y}_t = \mathbb{A}\tilde{y} + F(t), \quad \tilde{y}(0) = \tilde{y}_0, \quad (5.3)$$

where $F = (f_1, f_2; 0, 0) \in C^1([0, T]; Y)$. Therefore by Corollary 2.5[35, p.107] for any $\tilde{y}_0 \in \mathcal{D}(\mathbb{A})$ there exists a strong solution \tilde{y} to (5.3) in Y . This solution possesses the properties

$$\tilde{y} \in C((0, T); Y) \cap C^1((0, T); \mathcal{D}(\mathbb{A})')$$

and satisfies the relation

$$\|\tilde{y}(t)\|_Y^2 = \|\tilde{y}(0)\|_Y^2 + \int_0^t (F(\tau), \tilde{y}(\tau))_Y^2 d\tau. \quad (5.4)$$

Now we can return to the original variable $y(t) = (\phi(t) \equiv \tilde{\phi}(t) + \eta(t), \psi(t); u(t), v(t))$ and show that (5.4) can be written in the following way

$$\|y(t)\|_Y^2 = \|y(0)\|_Y^2 - 2U \int_0^t (\bar{u}_x(\tau), \gamma[\psi(\tau)])_{L_2(\Omega)} d\tau, \quad (5.5)$$

provided $\bar{u} \in C^2([0, T]; H_0^2(\Omega))$ and $\tilde{y}_0 \in \mathcal{D}(\mathbb{A})$.

The integral term in (5.5) can be estimated as follows:

$$\int_0^t | \langle \bar{u}_x, \gamma[\psi] \rangle | \leq c_0 \int_0^t \left[\|w\|_{H^2(\Omega)}^2 d\tau + \|\gamma[\psi]\|_{H^{-1/2}(\Omega)}^2 \right] d\tau \quad (5.6)$$

One can see that the estimate in (3.4) can be written with the constant C_T which is uniform at any interval, i.e. in the form

$$\int_0^t \|\gamma[\psi](\tau)\|_{H^{-1/2}(\mathbb{R}^2)}^2 d\tau \leq C_T \left(E_{fl}(0) + \int_0^t \|\partial_\nu \phi(\tau)\|^2 d\tau \right)$$

which holds for every $t \in [0, T]$. Since in our case $\partial_\nu \phi = -(v + U \partial_x \bar{u}) \cdot \mathbf{1}_\Omega(\mathbf{x})$, we have that

$$\int_0^t \|\gamma[\psi](\tau)\|_{H^{-1/2}(\mathbb{R}^2)}^2 d\tau \leq C_T \left(\|y_0\|_Y^2 + \int_0^t [\|v(\tau)\|^2 + \|\bar{u}(\tau)\|_{H^2(\Omega)}^2] d\tau \right)$$

for every $t \in [0, T]$. Therefore (5.5) and (5.6) yield

$$\|y(t)\|_Y^2 \leq C_T \left(\|y_0\|_Y + \int_0^t \|\bar{u}(\tau)\|_{H^2(\Omega)}^2 d\tau + \int_0^t \|y(\tau)\|_Y^2 d\tau \right)$$

for every $t \in [0, T]$, where $y_0 \in Y_1$ and $\bar{u} \in C^2([0, T]; H_0^2(\Omega))$. Now we can extend this inequality by continuity for all $y_0 \in Y$ and $w \in C([0, T]; H_0^2(\Omega))$ to obtain (3.27).

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